Deformations of trianalytic subvarieties of hyperkähler manifolds.

Misha Verbitsky,[1]

verbit@thelema.dnttm.rssi.ru, verbit@math.ias.edu

Let M be a compact complex manifold equipped with a hyperkähler metric, and X be a closed complex analytic subvariety of M. In alg-geom 9403006, we proved that X is **trianalytic** (i. e., complex analytic with respect to all complex structures induced by the hyperkähler structure), provided that M is generic in its deformation class. Here we study the complex analytic deformations of trianalytic subvarieties. We prove that all deformations of X are trianalytic and naturally isomorphic to X as complex analytic varieties. We show that this isomorphism is compatible with the metric induced from M. Also, we prove that the Douady space of complex analytic deformations of X in M is equipped with a natural hyperkähler structure.

Contents

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1 Introduction.

1.1 An overview

This is the third article studying closed complex analytic subvarieties of compact holomorphically symplectic[1] Kähler manifolds. In the first article in series ([V-I]), we proved that, when a holomorphically symplectic manifold M is generic in its deformation class, all subvarieties $X \subset M$ are also holomorphically symplectic, i. e. restriction of holomorphic symplectic form to nonsingular strata of X is non-degenerate. In the second article [V-II], we obtained a more precise result about the structure of such X, which is related to the hyperkähler structure of M.

By Yau's proof of Calabi conjecture, M admits a natural hyperkähler metric (Proposition 2.4; for a definition of hyperkähler manifold, see Definition 2.1). A hyperkähler structure (which is essentially a quaternion action in the tangent bundle to M) gives a rise to a whole family of complex structure on M, parametrized by $\mathbb{C}P^1$. These complex structures are called **complex structures induced by the hyperkähler structure** (Definition 2.2). Denote the set of all induced complex structures by $\mathcal{R}_M$. A closed subset $X \subset M$ is called **trianalytic** if X is complex analytic with respect to all induced complex structures $L \in \mathcal{R}_M$ (Definition 2.18). For an induced complex structure L, we denote by (M, L) the M considered as a complex manifold, with complex structure L. In [V-II], we proved that for all $L \in \mathcal{R}_M$, with exception of may be a countable set, all complex analytic subsets of (M, L) are trianalytic (Corollary 2.21).

Unlike the second article [V-II], which supersedes results of the first [V-I], the present one (the third) elaborates on the results of the second. We start where [V-II] left. Consider a complex subvariety $X \subset (M, L)$ which happens to be trianalytic. Such subvarieties are called **complex analytic subvarieties of trianalytic type**. We describe the deformations of X in (M, L) and the Douady space of such deformations.[2]

In [V-II], we gave a simple cohomological criterion of trianaliticity, for arbitrary complex analytic subvariety $X \subset (M, L)$ (see Theorem 2.19 of this article). Theorem 2.19 immediately implies that a complex analytic

[1] Holomorphically symplectic means "equipped with a holomorphic symplectic form". See Definition 2.3 for details.

[2] The Douady space [D] is the same as Chow scheme, but in complex analytic situation.

deformation of a trianalytic subvariety is again trianalytic. The main result of this article is the following theorem.

Theorem 1.1: Let M be a hyperkähler manifold, L an induced complex structure, and $X, X' \subset (M, L)$ be closed complex analytic subvarieties in the same deformation class. Assume that X is trianalytic.[3] Consider X, X' as Kähler subvarieties of (M, L), with Kähler metric induced from M. Then

(i) X, X' are naturally isomorphic as complex varieties, and this isomorphism is compatible with the Kähler metric.

(ii) The embedding $X \overset{i}{\hookrightarrow} M$ is compatible with metrics given by geodesics.

(iii) Also, every map $X \xrightarrow{\varphi} (M, L)$ is complex analytic, provided that φ is a deformation of i in the space of isometries $X \longrightarrow M$.

Caution The varieties X, X' need not to be non-singular.

Theorem 1.1 (i) follows from Theorem 9.1 and Theorem 9.2. Theorem 1.1 (ii) follows from Corollary 5.4, and (iii) from Proposition 7.5.

As a corollary, we obtain the following interesting result (Theorem 10.5).

Theorem 1.2: Let M be a compact holomorphically symplectic Kähler manifold, and $X \subset M$ a closed complex analytic subvariety. Consider the Douady space $D_M(X)$ of deformations of X inside M. Then $D_M(X)$ is compact and is equipped with a natural hyperkähler structure, in the sense of [V-bun],[4] (see Definition 10.1).

Remark 1.3: This gives a new set of examples of hyperkähler varieties, in addition to those produced by [V-bun]. The varieties obtained through [V-bun] are usually not compact; on contrary, our new examples are compact.

[3] Then the subvariety X' is also trianalytic, as implied by Theorem 2.19.

[4] If $D_M(X)$ is non-singular, this means exactly that $D_M(X)$ is hyperkähler. In singular case, there is no satisfactory definition of a hyperkähler structure. The definition of [V-bun] is a palliative, which is probably much stronger that the correct definition. See [S], [D] for alternative definitions.

The proof of Theorem 1.1 is based on the following argument. First, we show that every trianalytic submanifold $X \subset M$ (not necessarily closed) is **totally geodesic** in M (Corollary 5.4). [5] A simple geometric argument shows that a family $\mathcal{X}$ of totally geodesic submanifolds admits a natural **connection** (Definition 4.6, Proposition 4.7). This connection is compatible with the holomorphic structure if the family $\mathcal{X}$ is holomorphic (Proposition 4.7).

For M a compact hyperkähler variety, I an induced complex structure and X a trianalytic subvariety of M, we study the Douady space $D(X,I)$ which classifies complex analytic deformations of X in (M,I), where (M,I) is M considered as a complex manifold. We prove that the real analytic subvariety $D(X)$ underlying $D(X,I)$ does not depend from the choice of induced complex structure. This endows $D(X)$ with a 2-dimensional sphere of complex structures, which induce quaternionic action in the Zariski tangent space. A real analytic variety with such a system of complex structures is called **hypercomplex** (Definition 7.3).

We return to the study of the families of trianalytic subvarieties. As we have shown, the base of the universal family is hypercomplex. The fibers are trianalytic, and therefore, hypercomplex as well. The natural connection in such a family is compatible with the quaternionic action, because this connection is **holomorphic** with respect to each of induced complex structures. Thus, a curvature of this connection is $\mathbb{H}^*$-invariant (Lemma 8.7). On the other hand, the curvature lies in the tensor product of three representations of $\mathbb{H}^*$ of weight one. A trivial linear algebra argument shows that such a tensor product does not contain non-trivial $\mathbb{H}^*$-invariant vectors. This proves that the natural connection in the family of trianalytic subvarieties is **flat** (Theorem 8.6).

Let $\mathcal{X} \xrightarrow{\pi} S$ be a family of closed trianalytic subvarieties in a compact hyperkähler manifold M. Let X_s^{ns} be the set of non-singular points in $X_s := \pi^{-1}(s)$, $s \in S$. The submanifold X_s^{ns} is totally geodesic. Thus, the family $\mathcal{X}$ is equipped with a natural connection. For every two points $s_1, s_2 \in S$, we show that this connection might be integrated to an isomorphism of hyperkähler manifolds $\Psi_{s_1}^{s_2} : X_{s_1}^{ns} \longrightarrow X_{s_2}^{ns}$ (Theorem 9.1). Since X_s^{ns} is totally geodesic in M, its completion as a metric space is naturally isomorphic to a closure X_s of X_s^{ns} in M. Since the map $\Psi_{s_1}^{s_2} : X_{s_1}^{ns} \longrightarrow X_{s_2}^{ns}$ is an isometry, it is naturally extended to an isomorphism of metric spaces $\overline{\Psi}_{s_1}^{s_2} : X_{s_1}^{ns} \longrightarrow X_{s_2}^{ns}$ (Theorem 9.1). From construction it follows that $\Psi_{s_1}^{s_2}$

[5] **Completely geodesic** means that geodesics in X are also geodesics in M.

is compatible with each of induced holomorphic structures. We extend this assertion to $\overline{\Psi}_{s_1}^{s_2}$. This is done in two steps. A general type argument shows that a homeomorphism of complex varieties is bimeromorphic if it is holomorphic in a dense open set. The leap from bimeromorphic to holomorphic is done via a convoluted algebro-geometric argument involving normalization and finite unramified maps.

We finish this paper with the definition of singular hyperkähler varieties. We cite the previously known examples of singular hyperkähler varieties and then show that constructed above maps $\overline{\Psi}_{s_1}^{s_2} : X_{s_1}^{ns} \longrightarrow X_{s_2}^{ns}$ are isomorphisms of hyperkähler varieties. The Douady space of trianalytic subvarieties of a compact hyperkähler manifold is proven to be hyperkähler.

1.2 Contents

- Section 1 is an introduction, independent from the rest of this paper.
- Section 2 is a compendium of results pertaining to hyperkähler geometry and Yang–Mills theory. We define hyperkähler manifolds, trianalytic subvarieties, hyperholomorphic bundles, and cite their most basic properties. A reader acquainted with a literature may skip this section.
- Section 3 illustrates our results with a simplicistic example of deformations of a smooth trianalytic subvariety. This section is also independent from the rest, and its results are further superseded by more general statements. It is safe to skip this section too.
- Section 4 contains a study of totally geodesic submanifolds and their deformations. Using an easy geometric argument, we show that a family of totally geodesic submanifolds is equipped with a natural connection (Proposition 4.7). We study this connection and show that, under certain additional assumptions, this connection induces isometry (metric equivalence) of the fibers of this family. This section does not use results of hyperkähler geometry.
- In Section 5, we prove that hyperkähler submanifolds are always totally geodesic. We use arguments from Yang–Mills theory and the theory of hyperholomorphic bundles ([V-bun]).
- Section 6 shows that the deformational results of Section 4 can be applied to the deformations of hyperkähler submanifolds, not necessary closed. Again, we use arguments from the theory of hyperholomorphic

bundles. In Appendix to this section, we prove equivalence of real analytic structures induced on a hyperkähler manifold by different induced complex structures. The proof is based on twistor geometry ([HKLR], [KV]).

- Section 7 studies the Douady deformation space $D(X)$ of trianalytic subvariety X of a compact hyperkähler manifold M. A general argument of Kähler geometry shows that the Douady space is compact ([Li]). Using this, we prove that the underlying real analytic variety does not depend on the complex structure. As an application, we obtain that the Douady space is **hypercomplex** (Definition 7.3), i. e., has an integrable quaternionic action in the tangent space. In the Appendix to this section, we give an independent proof of the compactness of the Douady space. We describe the Douady space in terms of the space of isometric embeddings from X to M. The Appendix is based on Wirtinger's inequality (Theorem 2.26). The body of this section depends only on the result of the Appendix to Section 6 (the equivalence of induced real analytic structures on a given hyperkähler manifold).

- In Section 8, we define a curvature of a connection in a family of manifolds. We consider the natural connection on a family of trianalytic subvarieties of a compact hyperkähler manifolds. Using the results of Section 7 (hypercomplex structure on the Douady space), we show that this connection is **flat** i. e. its curvature is zero. This result is not used anywhere outside of Section 10 (and even there, we don't really need it). However, the flatness of the connection shows that the natural isomorphism $\Psi^{s_1}_{s_2}$ of the fibers of a family $\mathcal{X} \longrightarrow S$ of trianalytic subvarieties is independent from the choice of a path $\gamma: [0,1] \longrightarrow S$ in its homotopy class.

- Section 9 deals with the map $\Psi^{s_1}_{s_2}: X_{s_1} \longrightarrow X_{s_2}$ of fibers of a family of trianalytic subvarieties of M, obtained by integrating the natural connection. This map is by construction a homeomorphism and is holomorphic outside of singularities, with respect to each of induced complex structures. Using general argument, we show that $\Psi^{s_1}_{s_2}$ is bimeromorphic and induces an isomorphism of normalizations of X_{s_1}, X_{s_2}. Then, we apply the knowledge that, outside of singularities, X_{s_i} are totally geodesic in M. This allows us to show that $\Psi^{s_1}_{s_2}$ induces an isomorphism on the Zariski tangent spaces. An algebro-geometric argument shows that these properties are sufficient to prove that $\Psi^{s_1}_{s_2}$:

$X_{s_1} \longrightarrow X_{s_2}$ is a complex analytic isomorphism, with respect to each of induced complex structures.

In Section 9, we use the results of Section 4 (structure of deformations of totally geodesic submanifolds) and Section 6 (that these results may be applied to the deformations of trianalytic submanifolds).

- In Section 10, we give a definition of singular hyperkähler varieties. After a short discussion, we give examples of hyperkähler varieties, stemming from the theory of hyperholomorphic bundles ([V-bun]). All trianalytic subvarieties of hyperkähler manifolds are hyperkähler varieties, which is clear from the definition. Also, the map $\Psi_{s_2}^{s_1} : X_{s_1} \longrightarrow X_{s_2}$ constructed in Section 9, is an isomorphism of hyperkähler varieties. We show that the Douady space of deformations of a trianalytic subvariety is also a hyperkähler variety.

2 Basic definitions and results.

This section used mainly for reference, and contains a compilation of results and definitions from literature. An impatient reader is advised to skip it and proceed to the next section.

2.1 Hyperkähler manifolds

This subsection contains a compression of the basic and most known results and definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Beau].

Definition 2.1: ([Bes]) A **hyperkähler manifold** is a Riemannian manifold M endowed with three complex structures I, J and K, such that the following holds.

(i) the metric on M is Kähler with respect to these complex structures and

(ii) I, J and K, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, hyperkähler manifold has the natural action of quaternion algebra $\mathbb{H}$ in its real tangent bundle TM. Therefore its complex dimension is even. For each quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding automorphism of TM is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

Definition 2.2: Let M be a hyperkähler manifold, L a quaternion satisfying $L^2 = -1$. The corresponding complex structure on M is called **an induced complex structure**. The M considered as a complex manifold is denoted by (M, L).

Definition 2.3: Let M be a complex manifold and Ω a closed holomorphic 2-form over M such that $\Omega^n = \Omega \wedge \Omega \wedge ...$, is a nowhere degenerate section of a canonical class of M ($2n = dim_{\mathbb{C}}(M)$). Then M is called **holomorphically symplectic**.

Let M be a hyperkähler manifold; denote the Riemannian form on M by $< \cdot, \cdot >$. Let the form $\omega_I := < I(\cdot), \cdot >$ be the usual Kähler form which is closed and parallel (with respect to the Levi-Civitta connection). Analogously defined forms ω_J and ω_K are also closed and parallel.

A simple linear algebraic consideration ([Bes]) shows that the form $\Omega := \omega_J + \sqrt{-1}\omega_K$ is of type $(2,0)$ and, being closed, this form is also holomorphic. Also, the form Ω is nowhere degenerate, as another linear algebraic argument shows. It is called **the canonical holomorphic symplectic form of a manifold M**. Thus, for each hyperkähler manifold M, and induced complex structure L, the underlying complex manifold (M, L) is holomorphically symplectic. The converse assertion is also true:

Proposition 2.4: ([Beau], [Bes]) Let M be a compact holomorphically symplectic Kähler manifold with the holomorphic symplectic form Ω, a Kähler class $[\omega] \in H^{1,1}(M)$ and a complex structure I. There is a unique hyperkähler structure $(I, J, K, (\cdot, \cdot))$ over M such that the cohomology class of the symplectic form $\omega_I = (\cdot, I\cdot)$ is equal to $[\omega]$ and the canonical symplectic form $\omega_J + \sqrt{-1}\,\omega_K$ is equal to Ω.

Proposition 2.4 immediately follows from the conjecture of Calabi, proven by Yau ([Y]). ∎

2.2 Hyperholomorphic bundles

This subsection contains several versions of a definition of hyperholomorphic connection in a complex vector bundle over a hyperkähler manifold. We follow [V-bun].

Let M be a hyperkähler manifold. We identify the group $SU(2)$ with the group of unitary quaternions. This gives a canonical action of $SU(2)$ on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of $SU(2)$ on the bundle of differential forms. Of a special interest to us are those forms which are $SU(2)$-invariant.

Lemma 2.5: Let ω be a differential form over a hyperkähler manifold M. The form ω is $SU(2)$-invariant if and only if it is of Hodge type (p,p) with respect to all induced complex structures on M.

Proof: This is [V-bun], Proposition 1.2. ∎

Further in this article, we use the following statement.

Lemma 2.6: The action of $SU(2)$ on differential forms commutes with the Laplacian.

Proof: This is Proposition 1.1 of [V-bun]. ∎

Thus, for compact M, we may speak of the natural action of $SU(2)$ in cohomology.

Let B be a holomorphic vector bundle over a complex manifold M, ∇ a connection in B and $\Theta \in \Lambda^2 \otimes End(B)$ be its curvature. This connection is called **compatible with a holomorphic structure** if $\nabla_X(\zeta) = 0$ for any holomorphic section ζ and any antiholomorphic tangent vector X. If there exist a holomorphic structure compatible with the given Hermitian connection then this connection is called **integrable**.

One can define a **Hodge decomposition** in the space of differential forms with coefficients in any complex bundle, in particular, $End(B)$.

Theorem 2.7: Let ∇ be a Hermitian connection in a complex vector bundle B over a complex manifold. Then ∇ is integrable if and only if $\Theta \in \Lambda^{1,1}(M, \operatorname{End}(B))$, where $\Lambda^{1,1}(M, \operatorname{End}(B))$ denotes the forms of Hodge type (1,1). Also, the holomorphic structure compatible with ∇ is unique.

Proof: This is Proposition 4.17 of [Ko], Chapter I. ∎

Definition 2.8: Let B be a Hermitian vector bundle with a connection ∇ over a hyperkähler manifold M. Then ∇ is called **hyperholomorphic** if ∇ is integrable with respect to each of the complex structures induced by the hyperkähler structure.

As follows from Theorem 2.7, ∇ is hyperholomorphic if and only if its curvature Θ is of Hodge type (1,1) with respect to any of complex structures induced by a hyperkähler structure.

As follows from Lemma 2.5, ∇ is hyperholomorphic if and only if Θ is a $SU(2)$-invariant differential form.

Example 2.9: (Examples of hyperholomorphic bundles)

(i) Let M be a hyperkähler manifold, TM its tangent bundle equipped with Levi–Civita connection ∇. Then ∇ is integrable with respect to each induced complex structure, and hence, Yang–Mills.

(ii) For B a hyperholomorphic bundle, all its tensor powers are also hyperholomorphic.

(iii) Thus, the bundles of differential forms on a hyperkähler manifold are also hyperholomorphic.

2.3 Stable bundles and Yang–Mills connections.

This subsection is a compendium of the most basic results and definitions from the Yang–Mills theory over Kähler manifolds, concluding in the fundamental theorem of Uhlenbeck–Yau [UY].

Definition 2.10: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. We define $\deg(F)$ as

$$\deg(F) = \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}$$

and $\operatorname{slope}(F)$ as

$$\operatorname{slope}(F) = \frac{1}{\operatorname{rank}(F)} \cdot \deg(F).$$

The number $\operatorname{slope}(F)$ depends only on a cohomology class of $c_1(F)$.

Let F be a coherent sheaf on M and $F' \subset F$ its proper subsheaf. Then F' is called **destabilizing subsheaf** if $\operatorname{slope}(F') \geqslant \operatorname{slope}(F)$

A holomorphic vector bundle B is called **stable** if it has no destabilizing subsheaves.

Later on, we usually consider the bundles B with $deg(B) = 0$.

Let M be a Kähler manifold with a Kähler form ω. For differential forms with coefficients in any vector bundle there is a Hodge operator $L: \eta \longrightarrow \omega \wedge \eta$. There is also a fiberwise-adjoint Hodge operator Λ (see [GH]).

Definition 2.11: Let B be a holomorphic bundle over a Kähler manifold M with a holomorphic Hermitian connection ∇ and a curvature $\Theta \in \Lambda^{1,1} \otimes End(B)$. The Hermitian metric on B and the connection ∇ defined by this metric are called **Yang-Mills** if

$$\Lambda(\Theta) = constant \cdot \operatorname{Id}\Big|_B,$$

where Λ is a Hodge operator and $\operatorname{Id}\Big|_B$ is the identity endomorphism which is a section of $End(B)$.

Further on, we consider only these Yang–Mills connections for which this constant is zero.

A holomorphic bundle is called **indecomposable** if it cannot be decomposed onto a direct sum of two or more holomorphic bundles.

The following fundamental theorem provides examples of Yang--Mills bundles.

Theorem 2.12: (Uhlenbeck-Yau) Let B be an indecomposable holomorphic bundle over a compact Kähler manifold. Then B admits a Hermitian Yang-Mills connection if and only if it is stable, and this connection is unique.

Proof: [UY]. ■

Proposition 2.13: Let M be a hyperkähler manifold, L an induced complex structure and B be a complex vector bundle over (M, L). Then every hyperholomorphic connection ∇ in B is Yang-Mills and satisfies $\Lambda(\Theta) = 0$ where Θ is a curvature of ∇.

Proof: We use the definition of a hyperholomorphic connection as one with $SU(2)$-invariant curvature. Then Proposition 2.13 follows from the

Lemma 2.14: Let $\Theta \in \Lambda^2(M)$ be a $SU(2)$-invariant differential 2-form on M. Then $\Lambda_L(\Theta) = 0$ for each induced complex structure L.[1]

Proof: This is Lemma 2.1 of [V-bun]. ■

[1]By Λ_L we understand the Hodge operator Λ associated with the Kähler complex structure L.

Let M be a compact hyperkähler manifold, I an induced complex structure. For any stable holomorphic bundle on (M, I) there exists a unique Hermitian Yang-Mills connection which, for some bundles, turns out to be hyperholomorphic. It is possible to tell when this happens (though in the present paper we never use this knowledge).

Theorem 2.15: Let B be a stable holomorphic bundle over (M, I), where M is a hyperkähler manifold and I is an induced complex structure over M. Then B admits a compatible hyperholomorphic connection if and only if the first two Chern classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.[2]

Proof: This is Theorem 2.5 of [V-bun]. ■

2.4 Generic holomorphically symplectic manifolds

In this section, we define generic holomorphically symplectic manifolds. Such manifolds, as seen later (Corollary 2.21), admit a hyperkähler structure $\mathcal{H}$ such that every closed complex analytic subvariety is trianalytic with respect to $\mathcal{H}$ (for the definition of trianalytic subvarieities, see Definition 2.18). We follow [V-I] (see also [V-II]).

Let M be a compact holomorphically symplectic Kähler manifold. By Proposition 2.4, M has a unique hyperkähler structure with the same Kähler class and holomorphic symplectic form. Therefore one can without ambiguity speak about the action of $SU(2)$ on $H^*(M, \mathbb{R})$ (see Lemma 2.6). Of course, this action essentially depends on the choice of Kähler class.

Definition 2.16: Let $\omega \in H^{1,1}(M)$ be the Kähler class of a holomorphically symplectic manifold M. We say that ω **induces the $SU(2)$-action of general type** when all elements of the group

$$\bigoplus_p H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z}) \subset H^*(M)$$

are $SU(2)$-invariant. A holomorphically symplectic manifold M is called **of general type** if there exists a Kähler class on M which induces an $SU(2)$-action of general type.

[2]We use Lemma 2.6 to speak of action of $SU(2)$ in cohomology of M.

As Remark 2.22 below implies, holomorphically symplectic manifolds of general type have no Weil divisors. In particular, such manifolds are never algebraic.

Proposition 2.17: Let M be a hyperkähler manifold and S be the set of induced complex structures over M. Denote by $S_0 \subset S$ the set of $L \in S$ such that the natural Kähler metric on (M, L) induces the $SU(2)$ action of general type. Then S_0 is dense in S.

Proof: This is Proposition 2.2 from [V-II] ∎

One can easily deduce from the results in [Tod] and Proposition 2.17 that the set of points associated with holomorphically symplectic manifolds of general type is dense in the classifying space of holomorphically symplectic manifolds of Kähler type.

2.5 Trianalytic subvarieties in compact hyperkähler manifolds.

In this subsection, we give a definition and a few basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V-II].

Let M be a compact hyperkähler manifold, $\dim_{\mathbb{R}} M = 2m$.

Definition 2.18: Let $N \subset M$ be a closed subset of M. Then N is called **trianalytic** if N is a complex analytic subset of (M, L) for any induced complex structure L.

Throughout this paper, we implicitly assume that our trianalytic subvarieties are connected. Most results are trivially generalized to the general case.

Let I be an induced complex structure on M, and $N \subset (M, I)$ be a closed analytic subvariety of (M, I), $dim_{\mathbb{C}} N = n$. Denote by $[N] \in H_{2n}(M)$ the homology class represented by N. Let $\langle N \rangle \in H^{2m-2n}(M)$ denote the Poincare dual cohomology class. Recall that the hyperkähler structure induces the action of the group $SU(2)$ on the space $H^{2m-2n}(M)$.

Theorem 2.19: Assume that $\langle N \rangle \in H^{2m-2n}(M)$ is invariant with respect to the action of $SU(2)$ on $H^{2m-2n}(M)$. Then N is trianalytic.

Proof: This is Theorem 4.1 of [V-II] (see Subsection 2.7 for a sketch of a proof). ■

Remark 2.20: Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

Theorem 2.19 has the following immediate corollary:

Corollary 2.21: Let M be a compact holomorphically symplectic manifold of general type, $S \subset M$ be its complex analytic subvariety, and ω be a Kähler class which induces an $SU(2)$-action of general type. Consider the hyperkähler structure associated with ω by Proposition 2.4. Then S is trianalytic with respect to $\mathcal{H}$.

■

Remark 2.22: From Corollary 2.21 and Remark 2.20, it follows that a holomorphically symplectic manifold of general type has no closed complex analytic subvarieties of odd dimension; in particular, such a manifold has no divisors.

2.6 Wirtinger's inequality and its use in Kähler geometry

Let M be a compact Kähler manifold, $X \subset M$ a closed real analytic subvariety. In this section, we give criteria for X to be complex analytic, in terms of certain integrals. We follow [V-II] and [Sto].

Definition 2.23: Let H be an $\mathbb{R}$-linear space equipped with a positively defined scalar product, $\dim H = h$. The exterrior form $\mathrm{Vol} \in \Lambda^h(H)$ is called **a volume form** if the the standard hypercube with the side 1 has the volume 1 in the measure defined by Vol.

Clearly, the volume form is defined up to a sign. This sign is determined by the choice of orientation on H. In the same manner we define the top degree differential form Vol called **a volume form** on any oriented Riemannian manifold.

Let V be a Hermitian linear space, $W \subset V$ be a $\mathbb{R}$-linear subspace, $\dim_{\mathbb{R}} W = 2n$. Consider space $\Lambda^{2n}(W)$ of volume forms on M. Let ω be the imaginary part of the Hermitian form on V. Consider the vectors ω^n, $\mathrm{Vol} \in \Lambda^{2n}(W)$. Since Vol is non-zero, and $\Lambda^{2n}(W)$ is one-dimensional, we can speak of a fraction $\frac{\omega^n}{\mathrm{Vol}}$, which is a real number, defined up to a sign (the form Vol is defined up to a sign). Denote by Ξ_W the number $\left|\frac{\omega^n}{\mathrm{Vol}}\right|$.

Lemma 2.24: (Wirtinger's inequality) In these assumptions, $\Xi_W \leqslant 2^n$. Moreover, if $\Xi_W = 2^n$, then W is a complex subspace of V.

Proof: [Sto] page 7. ■

Let M be a Kähler manifold, $N \subset (M, I)$ be a closed real analytic subvariety of even dimension, $N_{ns} \subset N$ the set of non-singular points of N. For each $x \in N_{ns}$, consider $T_x N$ as a subspace of $T_x M$. Denote the corresponding number $\Xi_{T_x N}$ by $\Xi_L(x)$. The following statement is a direct consequence of Lemma 2.24:

Proposition 2.25: Let J be an induced complex structure. The subset $N \subset M$ is complex analytic if and only if

$$\forall x \in N_{ns} \quad \Xi_L(x) = 2^n.$$

■

Let $\operatorname{Vol} N_{ns}$ be the volume form of N_{ns}, taken with respect to the Riemannian form (see Definition 2.23).

Theorem 2.26: Let M be a Kähler manifold, $N \subset M$, $\dim N = 2n$ a closed real analytic subvariety, $N_{ns} \subset M$ the set of its non-singular points. Assume that the improper integrals $\int_{N_{ns}} \operatorname{Vol} N_{ns}$, $\int_{N_{ns}} \omega^n$ exist. Then

$$2^n \int_{N_{ns}} \operatorname{Vol} N_{ns} \geqslant \int_{N_{ns}} \omega^n$$

and the equality is reached if and only if N is complex analytic in M.

Proof: This is a direct consequence of Proposition 2.25 ■

We use the term "symplectic volume" for the number $\frac{1}{2^n} \int_{N_{ns}} \omega^n$ and "Riemannian volume" for $\int_{N_{ns}} \operatorname{Vol} N_{ns}$. Then, Theorem 2.26 might be rephrased in the form "a real analytic cycle is complex analytic if and only if its symplectic volume is equal to its Riemannian volume".

2.7 $SU(2)$-invariant cycles in cohomology and trianalytic subvarieties

This subsection consists of a sketch of a proof of Theorem 2.19. We follow [V-II].

For a Kähler manifold M, $m = dim_{\mathbb{C}} M$ and a form $\alpha \in H^{2p}(M, \mathbb{C})$, define

$$\deg(\alpha) := \int_M \omega^{m-p} \wedge \alpha,$$

where ω is the Kähler form.

Let M be a compact hyperkähler manifold, α a differential form. We denote by $\deg_L \alpha$ the degree associated with an induced complex structure L.

Proposition 2.27: Let M be a compact hyperkähler manifold and α be an $SU(2)$-invariant form of non-zero degree. Then the dimension of α is divisible by 4. Moreover,

$$\deg_I \alpha = \deg_{I'} \alpha,$$

for every pair of induced complex structures I, I'.

Proof: This is Proposition 4.5 of [V-II]. ■

Theorem 2.19 follows immediately from Proposition 2.27 and Theorem 2.26. By Theorem 2.26, a real analytic subvariety $N \subset M$ is trianalytic if and only if its symplectic volume, taken with respect to any of induced complex structures, is defined and equal to its Riemannian volume. In notation of Theorem 2.19, the symplectic volume of N taken with respect to L is equal to $\frac{1}{2^n} \deg_L \langle N \rangle$. From Proposition 2.27, we obtain that symplectic volume of N is the same for all induced complex structures. Since N is complex analytic with respect to I, the symplectic volume of N is equal to the Riemannian volume of N, again by Theorem 2.26. This proves Theorem 2.19.

3 Deformations of non-singular trianalytic subvarieties.

Let M be a holomorphically symplectic Kähler manifold, $X \subset M$ a closed complex analytic submanifold of trianalytic type. We study the deformations of X in M, in order to prove Theorem 1.1. It turns out that Theorem 1.1 is almost trivial in the case X non-singular. In this section, we give the proof of Theorem 1.1 (i) for non-singular X; the general case is proven independently. We hope that a simple argument will be insightful, even if we need to produce a separate proof for the general case. This section is perfectly safe to skip.

Remark 3.1: For a smooth trianalytic submanifold $X \subset M$, X is obviously hyperkähler, in a natural way. The quaternion action comes from quaternion action in TM, and the metric is induced from M too.

We start from the following simple, but important, lemma.

Lemma 3.2: Let M be a complex manifold equipped with a hyperkähler metric, $X \subset M$ a closed complex analytic submanifld of trianalytic type. Consider the restriction $TM\Big|_X$ of the tangent bundle to M on X. We equip $TM\big|_X$ with a connection ∇ coming from Levi-Civita connection in TM. Then $(TM\Big|_X, \nabla)$ is **hyperholomorphic**.[1]

Proof: Consider the natural action of $SU(2)$ on the space

$$\Lambda^2 \left(X, \operatorname{End}\left(TM\Big|_X\right)\right).$$

We need to show that the curvature Θ of ∇ is $SU(2)$-invariant. Let ∇_{LC} be the Levi–Civita connection on TM, and $(\nabla_{LC})^2 \in \Lambda^2(M, \operatorname{End}(TM))$ be its curvature. Clearly, Θ is a pull-back of $(\nabla_{LC})^2$ to X. Therefore, it suffices to show that $(\nabla_{LC})^2$ is $SU(2)$-invariant. This is Example 2.9. ∎

As a corollary, we obtain the following proposition.

Proposition 3.3: In assumptions of Lemma 3.2, let M, X be compact. Then the following statements are true.

(i) The bundle $TM\Big|_X$ is naturally isomorphic to the direct sum $TM\Big|_X \cong TX \oplus NX$, where N is the normal bundle to X inside M. This isomorphism is compatible with the natural connections and Hermitian metrics on $TM\Big|_X$, TX, NX.

(ii) NX is hyperholomorphic.

(iii) For each section γ of NX, γ is nowhere degenerate, and there is a natural decomposition

$$NX = \mathcal{O}_\gamma \oplus N_\gamma X, \tag{3.1}$$

[1]See Subsection 2.2 for the definition of **hyperholomorphic**. By Proposition 2.13, a hyperholomorphic connection is Yang-Mills.

where $\mathcal{O}_\gamma$ is a trivial sub-bundle of NX generated by γ, and $N_\gamma X$ its orthogonal complement. The decomposition (3.1) is compatible with connection.

(iv) In assumptions of (iii), consider the connection ∇_γ induced from NX to $\mathcal{O}_\gamma X$. Then is ∇_γ flat.

Remark 3.4: In fact, Proposition 3.3 (i)-(ii) holds true in assumptions of Lemma 3.2 for general, not necessarily compact, X and M (see Proposition 5.2, Proposition 5.1). However, the proof is easier in compact case.

Proof of Proposition 3.3. Consider the embedding

$$TX \hookrightarrow TM\Big|_X$$

of bundles with connection. By Example 2.9, TX is hyperholomorphic, and hence Yang–Mills. By Lemma 3.2, $TM\Big|_X$ is hyperholomorphic as well. We obtain that TX is a destabilizing subsheaf in TX. By Theorem 2.12, a Yang–Mills bundle is a direct sum of stable bundles.[2] Thus, TM is a direct sum of TX and its orthogonal complement NX (see Proposition 5.5 for details). This proves Proposition 3.3 (i). Since the curvature of $TM\Big|_X$ is decomposed onto a direct sum of the curvature of NX and the curvature of TX, the curvature of NX is $SU(2)$-invariant. This proves Proposition 3.3 (ii).

The condition (iii) follows from (ii) and Theorem 2.12, since every holomorphic section γ of a Yang–Mills bundle B of zero degree (such as hyperholomorphic bundles) spans a destabilizing subsheaf $\mathcal{O}_\gamma$. Since B is Yang–Mills, it is polystable; thus, $\mathcal{O}_\gamma$ is its direct summand. This proves Proposition 3.3 (iii). To prove (iv), we notice that $\mathcal{O}_\gamma$ is a trivial holomorphic bundle with Yang–Mills connection. By Theorem 2.12, Yang–Mills connection is unique, and therefore, $\mathcal{O}_\gamma$ is flat. Proposition 3.3 is proven. ∎

We recall the following general results of the theory of deformations of complex subvarieties. Let $X \subset M$ be a closed complex analytic subvariety of a compact complex manifold. The **Douady space** of deformations of X inside of M is defined ([D]). We denote the Douady space by $D_M X$, or

[2] Bundles which are direct sum of stable are called **polystable**.

sometimes by $D(X)$. By definition, $D_M X$ is a finite-dimensional complex analytic variety. The points of $D_M X$ are identified with the subvarieties $X' \subset M$, where X' is a deformation of X. Let $\gamma: \mathbb{R} \longrightarrow D_m X$ be a real analytic map. Assume that for $t = t_0$, $\gamma(t)$ is a smooth subvariety of M. In deformation theory, the differential $\frac{d\gamma}{dt}(t_0)$ is interpreted as a holomorphic section of the normal bundle $N\gamma(t_0)$. Thus, the Zariski tangent space $T_X D_M(X)$ is naturally embedded to the space of holomorphic sections of NX. The following proposition is an easy application of the Kodaira–Spencer theory.

Proposition 3.5: Let $X \subset M$ be a closed complex analytic submanifold of a compact complex manifold M, and $D_M X$ the corresponding Douady space. Let $\gamma: \mathbb{R} \longrightarrow D_M X$ be a real analytic map. Assume that for all $t_0 \in \mathbb{R}$, the subvariety $\gamma(t_0)$ is smooth. Assume also that the section

$$\nu_{t_0} = \frac{d\gamma}{dt}(t_0) \in \Gamma_{\gamma(t_0)}(N\gamma(t_0))$$

is nowhere degenerate. Finally, assume that ν_{t_0} splits from $N\gamma(t_0)$, i. e, there exists a decomposition of a holomorphic vector bundle

$$N\gamma(t_0) = \mathcal{O}_{\nu_{t_0}} \oplus N'$$

where $\mathcal{O}_{\nu_{t_0}}$ is the trivial subbundle of $N\gamma(t_0)$ generated by ν_{t_0}. Then the subvarieties $\gamma(t) \subset M$ are naturally isomorphic for all t.

Proof: Well known; see, for instance, [KS] ■

Proposition 3.6: In assumptions of Proposition 3.5, let M be Kähler. Consider the induced Hermitian structure on $N\gamma(t)$, for all $t \in \mathbb{R}$. Assume that for all $t \in \mathbb{R}$, the section

$$\nu_{t_0} = \frac{d\gamma}{dt}(t_0)$$

has constant length. Assume also that the orthogonal decomposition

$$TM\Big|_X = NX \oplus NX^\perp$$

is compatible with the Levi-Civita connection in $TM\Big|_X$. Consider the isomorphisms $\psi_{t_1,t_2}: \gamma(t_1) \longrightarrow \gamma(t_2)$ constructed in Proposition 3.5. Then the maps ψ_{t_1,t_2} are compatible with the Kähler metric induced from M.

Proof: The proof follows from Kodaira–Spencer construction; for a complete argument, see Proposition 4.8 (ii). ∎

The following theorem is the main result of this section.

Theorem 3.7: Let M be a compact holomorphically symplectic Kähler manifold, and $X \subset M$ a complex submanifold of trianalytic type. Let X' be a deformation of X in M. Then there exists a complex analytic isomorphism $\psi : X \longrightarrow X'$. Moreover, if the Kähler metric on M is hyperkähler, then $\psi : X \longrightarrow X'$ is compatible with the Kähler structure induced from M.

Proof: The deformations of X are infinitesimally classified by the sections of NX. Let γ be such a section. Applying Proposition 3.3, we obtain that assumptions of Proposition 3.5 and Proposition 3.6 hold for the deformations of X inside M. Now, Theorem 3.7 is directly implied by Proposition 3.5 and Proposition 3.6. ∎

Remark 3.8: Theorem 3.7 gives a proof of Theorem 1.1 (i), for the subvarieties $X \subset M$ which are smooth. We prove Theorem 3.7 for general subvarieties in Theorem 9.2.

4 Completely geodesic embeddings of Riemannian manifolds.

Before we proceed with the proof of Theorem 1.1, we have to prove a serie of preliminary results from deformation theory. The arguments of deformation theory are greatly simplified in the hyperkähler case, because hyperkähler embeddings are totally geodesic[1] (Corollary 5.4). We prove a number of simple statements from the deformation theory of totally geodesical embeddings of Kähler manifolds.

4.1 Completely geodesic submanifolds

Proposition 4.1: Let $X \stackrel{\varphi}{\hookrightarrow} M$ be an embedding of Riemannian manifolds (not necessarily compact) compatible with the Riemannian structure.[2] Then the following conditions are equivalent.

[1] For a definition of totally geodesical embeddings, see Definition 4.2.
[2] Such embeddings are called **Riemannian embeddings**.

(i) For every point $x \in X$, there exist a neighbourhod $U \ni x$ of x in X such that for all $x' \in U$ there is a geodesic in M going from $\varphi(x)$ to $\varphi(x')$ which lies in $\varphi(X) \subset M$.

(ii) Consider the Levi-Civita connection ∇ on TM, and restriction of ∇ to $TM\Big|_X$. Consider the orthogonal decomposition

$$TM\Big|_X = TX \oplus TX^\perp. \tag{4.1}$$

Then, this decomposition is preserved by the connection ∇.

Proof: Well known; see, for instance, [Bes]. ∎

Definition 4.2: Let $X \stackrel{i}{\hookrightarrow} M$ be a Riemannian embedding satisfying either of the conditions of Proposition 4.1. Then i is called **a totally geodesic embedding**, and the image $i(X) \subset M$ is called **a totally geodesic submanifold**.

Lemma 4.3: Let M be a Kähler manifold, $X \subset M$ a complex submanifold. If X is totally geodesic, then the decomposition (4.1) is compatible with the holomorphic structure on TX, $TM\Big|_X$.

Proof: Clear. ∎

4.2 Deformations of submanifolds

Let $\mathcal{X} \stackrel{\pi}{\longrightarrow} S$ be a family of complex manifolds equipped with a map $\mathcal{X} \stackrel{\varphi}{\longrightarrow} M$. Denote the pre-image $\pi^{-1}(s) \subset \mathcal{X}$ by X_s. Assume that for all $s \in S$ the restriction $\varphi_s: X_s \longrightarrow M$ of $\mathcal{X} \stackrel{\varphi}{\longrightarrow} M$ to $X_s \subset \mathcal{X}$ is a smooth embedding. Assume that for $s_0 \in S$, the image $\varphi_{s_0}(X_{s_0}) \subset M$ coinsides with X.

Definition 4.4: The collection of data

$$\left(\mathcal{X} \stackrel{\pi}{\longrightarrow} S, \varphi, \varphi_{s_0}: X_{s_0} \stackrel{\sim}{\longrightarrow} X\right)$$

is called **a family of submanifolds of** M, and **a family of deformations of** X. The same definition can be formulated for M, X, S, $\mathcal{X}$ real analytic; in this case, we speak of **family of real analytic submanifolds of** M. Also, the submanifolds might be replaced by subvarieties in order to obtain the definition of a family of subvarieties.

4.3 Section of a normal bundle arising from a deformation

Let M be a complex or real analytic manifold and

$$\left(\mathcal{X} \xrightarrow{\pi} S, \varphi: \mathcal{X} \longrightarrow M\right)$$

be a system of submanifolds. For each tangent vector $t \in T_{s_0}S$, deformation theory gives a canonical section η of the normal bundle NX_{s_0}. This section is holomorphic if we work in complex situation. We recall how this section is obtained.

For a sufficiently small neighbourhood of $x \in \mathcal{X}$, we can always find coordinates in X_s which analytically depend on s. For each point $x \in X_{s_0}$, denote by $x(s)$ the point of X_s with the same coordinates as x. Consider the vector $\widetilde{\eta}_x := \frac{dx(s)}{ds}(t) \in T_xM$, which is a derivative of $x(s)$ along t. The vector $\widetilde{\eta}_x$ obviously depends on the choice of coordinates in X_s. Let $\eta_x \in NX_{s_0}\Big|_x$ be the image of $\widetilde{\eta}_x$ under the natural map $T_xM \longrightarrow NX_{s_0}\Big|_x$. Clearly, η_x is independent from the choice of coordinates. Gluing η_x together, we obtain the canonical section $\eta \in NX_{s_0}$.

We need the following technical lemma in Section 6.

Lemma 4.5: Let M be a complex analytic manifold and

$$\left(\mathcal{X} \xrightarrow{\pi} S, \varphi: \mathcal{X} \longrightarrow M\right)$$

be a real analytic system of submanifolds. Assume that for all $s \in S$, the submanifold $\pi^{-1}(s) := X_s \subset M$ is complex analytic. Then, for all $t \in T_{s_0}S$, the corresponding section $\eta \in NX_{s_0}$ is holomorphic.

Proof: Shrinking $\mathcal{X}$ if necessary, we can find a system of holomorphic coordinates in X_s, $s \in S$ which depends smoothly upon s. Consider the section $\widetilde{\eta} \in TM\Big|_{X_{s_0}}$ obtained by deriving $x(s)$ along t as above. Since the map $x \longrightarrow x(s)$ is by construction holomorphic, the section $\widetilde{\eta}$ is also holomorphic. Then, η is holomorphic by construction. ∎

4.4 Connections in families of manifolds

Definition 4.6: Let $\pi: \mathcal{X} \longrightarrow S$ be a family of real analytic manifolds parametrized by S. Consider the bundles $T\mathcal{X}$, $N_\pi X$, where $N_\pi X$ is a fiberwise normal bundle to the fibers of π. There is a natural projection $p: T\mathcal{X} \longrightarrow N_\pi X$. The **connection** in $\mathcal{S}$ is a section of p, i. e. such an embedding $\nabla: N_\pi X \longrightarrow T\mathcal{X}$ that $p \circ s = \operatorname{Id}\Big|_{N_\pi X}$.

This definition is naturally adopted to the case of complex analytic family of manifolds. If the section $\nabla: N_\pi X \longrightarrow T\mathcal{X}$ is holomorphic, the connection is called **a holomorphic connection**.

This definition makes sense only when the base variety S is smooth. However, the connection in a family gives a connection on a pullback of this family, under all maps $s: S' \longrightarrow S$. Connections on the pullback families are naturally compatible. We shall sometimes speak of **connection** in a family where the base S is not smooth. This means that for all maps $s: S' \longrightarrow S$, where S' is smooth, the pullback family is equipped with a connection, and these connections are compatible. All the definitions and results (which we state and prove in the case of a smooth base) are naturally adopted to the case when the base S is singular.

Proposition 4.7: Let

$$\left(\mathcal{X} \xrightarrow{\pi} S, \varphi: \mathcal{X} \longrightarrow M\right)$$

be a family of submanifolds of a Riemannian manifold M. Assume that for all $s \in S$, the submanifold $X_s := \psi(\pi^{-1}(s)) \subset M$ is totally geodesic. Then the family $\mathcal{X}$ is equipped with a natural connection. Moreover, if M is Kähler and the family $\mathcal{X}$ is complex analytic, then the connection ∇ is holomorphic.

Proof: Let $T_\pi X \subset T\mathcal{X}$ be the bundle of vectors tangent to the fibers of π. To split the exact sequence

$$0 \longrightarrow T_\pi X \xrightarrow{i} T\mathcal{X} \longrightarrow T_\pi X \longrightarrow 0,$$

we have to construct a surjection

$$T\mathcal{X} \xrightarrow{p} T_\pi X \tag{4.2}$$

satisfying $i \circ p = \operatorname{Id}_{T_\pi X}$. Consider the orthogonal decomposition

$$\varphi^* TM = \varphi^* N_M X_t \oplus TX_t, \tag{4.3}$$

where $N_M X_t$ is the normal bundle to $\varphi(X_t)$ in M. This gives a natural epimorphism

$$\varphi^* TM \xrightarrow{e} T_\pi X.$$

Taking a composition of e with

$$d\varphi:\ T\mathcal{X} \longrightarrow \varphi^* TM,$$

we obtain a surjection p of 4.2.

In Kähler case, the decomposition (4.3) is holomorphic by Lemma 4.3. Thus, we constructed a connection which is holomorphic. ■

For each real analytic path $\gamma:\ [0,1] \longrightarrow S,\ t, t' \in [0,1]$, we may **integrate** the connection along γ. This notion is intuitively clear (and well known). We recall briefly the definition of the integral maps associated to connection, in order to fix the notation.

Restricting the family $\mathcal{X}$ to the image of γ in S does not change the result of integration. To simplify the exposition, we assume that S coinsides with the image $D \cong [0,1]$ of γ. In this case, the bundle $N_\pi X$ is naturally isomorphic to a pullback $\pi^* TD$. Denote by ν a canonical unit section of $\pi^* TD$ corresponding to the unit tangent field to $[0,1]$. Then, $\nabla\nu$ is a vector field in $T\mathcal{X}$. Integrating ν to a diffeomorphism, we obtain a map $\exp(t\nu)$ defined in an open subset of $\mathcal{X}$, for $t \in \mathbb{R}$. Clearly, $\exp(t\nu)$ maps the points of X_{t_1} to the points of X_{t_1+t}. The resulting diffeomorphism we denote by

$$\Psi^{t_1}_{t_1+t}:\ U_{t_1} \longrightarrow U_{t+t_1},$$

where U_{t_1} is an intersection of X_{t_1} with the domain of $\exp(t\nu)$. Clearly, the same definition applies to the arbitrary families of manifolds with connection. When $\mathcal{X}$ is a complex analytic family with a holomorphic connection, the diffeomorphisms $\Psi^t_{t'}$ are complex analytic.

4.5 Deformations of totally geodesic submanifolds (the main statement)

Let

$$\left(\mathcal{X} \xrightarrow{\pi} S, \varphi, \varphi_{s_0}:\ X_{s_0} \xrightarrow{\sim} X\right)$$

be a real analytic deformation of $X \subset M$, and $\gamma:\ [0,1] \longrightarrow S$ be a real analytic map. Slightly abusing the notation, we shall consider the fibers X_s of π as subvarieties in M. Assume that $X_{\gamma(t)}$ is a totally geodesic complex analytic submanifold of M, for all t. Let $\eta_t \in \Gamma\left(NX_{\gamma(t)}\right)$ be a section of the normal bundle to $X_{\gamma(t)}$ corresponding to the vector $\frac{d\gamma}{dt} \in T_{\gamma(t)}S$. Using the natural holomorphic embedding

$$NX_{\gamma(t)} \hookrightarrow TM\Big|_{X_{\gamma(t)}} \tag{4.4}$$

associated with a decomposition $TM\Big|_{X_{\gamma(t)}} = NX_{\gamma(t)} \oplus TX_{\gamma(t)}$, we may consider η_t as a section of $TM\Big|_{X_{\gamma(t)}}$.

Proposition 4.8: Let t, $t' \in [0,1]$. Let $\Psi^t_{t'}: U_t \longrightarrow U_{t'}$ be the map of Subsection 4.4 integrating the connection ∇ of Proposition 4.7, where U_t, $U_{t'}$ are open subsets of $X_{\gamma(t)}$, $X_{\gamma(t')}$ defined in Subsection 4.4. Assume that for all t the section $\eta_t \in TM\Big|_{X_{\gamma(t)}}$ is parallel with respect to the natural connection on $TM\Big|_{X_{\gamma(t)}}$ obtained from Levi–Civita on TM. Assume also that the Riemannian form on M is real analytic.[3] Then $\Psi^t_{t'}$ satisfies the following conditions.

(i) $\Psi^t_{t'}$ is compatible with a Riemannian structure on U_t, $U_{t'}$.

(ii) Assume also that M admits a smooth Riemannian compactification $\overline{M}$ such that the family $\mathcal{X}$ admits a compactification in $\overline{M}$ (such is the case when M and $\mathcal{X}$ are quasiprojective, and φ is algebraic). Then the maps $\Psi^t_{t'}: U_t \longrightarrow U_{t'}$ of (i) can be found in such a way that U_t is the set of non-singular points of $X_{\gamma(t)}$.

(iii) In assumptions of (ii), we can extend $\Psi^t_{t'}$ to an isomorphism of metric spaces

$$\overline{\Psi}^t_{t'}: \overline{X}_{\gamma(t)} \longrightarrow \overline{X}_{\gamma(t')},$$

where $\overline{X}_{\gamma(t)}$, $\overline{X}_{\gamma'(t)}$ are closures of $X_{\gamma(t)}$, $X_{\gamma'(t)}$ in $\overline{M}$, with induced metrics.

The next subsection is taken by the proof of Proposition 4.8.

4.6 Deformations of totally geodesic submanifolds (the proofs)

The statement (i) is sufficient to prove in a small neighbourhood, by analytic continuation. Thus, we may pick an open subset $U \in \mathcal{X}$ in such a way that the restriction of $\varphi: \mathcal{X} \longrightarrow M$ to

$$\mathcal{X}' := \pi^{-1}(\gamma([0,1]) \cap U$$

[3]This assumption is extraneous; we use it to simplify the exposition. For the case we are interested in (M a hyperkähler manifold) the Riemannian form is real analytic by Corollary 6.6.

is an embedding. Shrinking $\mathcal{X}$ to $\mathcal{X}'$ and pulling from M the Riemannian metric, we see that Proposition 4.8 (i) is implied by the following lemma.

Lemma 4.9: Let $\mathcal{X} \xrightarrow{\pi} [0,1]$ be a real analytic family of manifolds, equipped with a Riemannian metrics. Let $T_\pi X \subset T\mathcal{X}$ be the relative tangent bundle consisiting of all vectors tangent to the fibers of π, and $N_\pi X = T\mathcal{X}/T_\pi X$ be the normal bundle to the fibration π. Assume that the fibers $\pi^{-1}(t) \subset \mathcal{X}$ are totally geodesic in $\mathcal{X}$. Consider the connection $\nabla : N_\pi X \longrightarrow T\mathcal{X}$ of Proposition 4.7. Let $\Psi_{t'}^t : U_t \longrightarrow U_{t'}$ be the maps obtained by integrating the connection as in Subsection 4.4 and $\eta_t \in N_\pi X$ be the normal fields arising from the deformation theory. Assume that η_t's are parallel with respect to the natural connection on $N_\pi X$. Then the maps $\Psi_{t'}^t$ are compatible with Riemannian metrics.

Proof: Let $\eta \in T\mathcal{X}$ be the tangent vector field obtained by gluing all η_t together. Then Ψ_t can be considered as integral map of this vector field. Thus, to prove that Ψ_t is compatible with the Riemannian structure, we have to show that η is **a Killing vector field.**[4] Denote by $\nabla_x : T\mathcal{X} \longrightarrow T\mathcal{X}$ the action of covariant derivative along the vector field x. By [Bes], Theorem 1.81, to prove that η is Killing it suffices to prove that for all fields $a, b \in T\mathcal{X}$, we have

$$(\nabla_a \eta, b) + (\nabla_b \eta, a) = 0, \tag{4.5}$$

where $(\cdot,\cdot)$ is the Riemannian form. Take coordinates $(x_0, x_1, ... x_n)$ on $\mathcal{X}$ in such a way that x_0 comes from a projection $\pi : \mathcal{X} \longrightarrow [0,1]$ and $x_1, ... x_n$ are coordinates along the fibers. Let $\frac{d}{dx_i}$ be the corresponding vector fields. To prove that η is Killing it suffices to check (4.5) for a, b coordinate vector fields. Since η is parallel along X_t, we have $\nabla_{\frac{d}{dx_i}} \eta = 0$ for $i = 1, ..., n$. Thus, it suffices to prove (4.5) in case $a = \frac{d}{dx_0}$. For appropriate choice of coordinates, $\frac{d}{dx_0} = \eta$; thus, (4.5) is implied by

$$\left(\nabla_\eta \eta, \frac{d}{dx_i}\right) = 0, \quad i = 0, \ldots n. \tag{4.6}$$

Since Levi-Civita connection is compatible with the metrics, we have

$$D_\eta \left(\eta, \frac{d}{dx_i}\right) = \left(\nabla_\eta \eta, \frac{d}{dx_i}\right) + \left(\eta, \nabla_\eta \frac{d}{dx_i}\right), \tag{4.7}$$

[4] A Killing field is a vector field which integrates to a diffeomorphism which is compatible with a Riemannian metric.

where D_η is the usual (directional) derivative along η. For $i=0$, this gives

$$D_\eta(\eta,\eta) = (\nabla_\eta\eta,\eta) + (\eta,\nabla_\eta\eta) = 0$$

(η is parallel, and hence has constant length). This proves (4.6) for $i=0$. Since $\frac{d}{dx_i}$ are coordinate vector fields they commute, and therefore, $\nabla_\eta\frac{d}{dx_i} = -\nabla_{\frac{d}{dx_i}}\eta = 0$ for $i>0$. Thus, the equation (4.7) for $i>0$ is reduced to

$$D_\eta\left(\eta,\frac{d}{dx_i}\right) = \left(\nabla_\eta\eta,\frac{d}{dx_i}\right).$$

Since the vector fields $\frac{d}{dx_i}$, $i>0$ are tangent to the fibration π, the function $(\eta,\frac{d}{dx_i})$ is identically zero. This proves (4.6) for $i>0$. Lemma 4.9 and consequently Proposition 4.8 (i) is proven. ∎

To prove Proposition 4.8 (ii) we have to show that the connection of Proposition 4.7 can be integrated for all smooth points of $\mathcal{X}$. We give a sketch of a simple geometric argument. By (i), the maps Ψ_t are isometries. Thus, the distance from the given point to the singular set of X_t is invariant under the maps Ψ_t. To integrate the connection, we write a tangent vector field which we subsequently integrate. Since the normal field η_t is parallel, this tangent field is uniformly bounded. On a certain distance from the singular set $Sing(X_t)$, depending on this uniform bound, the connection might be always integrated. In more precise terms, for all $\varepsilon>0$, all $x\in X_t$, with the distance between x and $Sing(X_t)$ no less than ε, there exist $\delta>0$ and a map $\Psi^t_{t+\delta}: \ X_t \longrightarrow X_{t+\delta}$ defined in a neighbourhood of x which integrates the connection. From this statement and Proposition 4.8 (i), Proposition 4.8 (ii) follows directly.

Finally, to prove Proposition 4.8 (iii) we notice that X_t is totally geodesic in M for all t. Thus, the completion of X_t as a metric space coinsides with the closure of X_t in $\overline{M}$. Every isometry of metric spaces extends to a completion, and thus, $\Psi^t_{t'}$ extends to a closure of X_t, $X_{t'}$ in $\overline{M}$. ∎

5 Hyperholomorphic bundles and totally geodesical embeddings.

In this section, we prove that hyperkähler embeddings are totally geodesic.

5.1 Hyperholomorphic structure on the normal bundle

Let M be a hyperkähler manifold, not necessary compact, and B a vector bundle. Recall that in Section 2, we defined **hyperholomorphic connections** in B (Definition 2.8). These are connections ∇ such that the curvature $\Theta := \nabla^2 \in \Lambda^2(M, End(B))$ is an $SU(2)$-invariant 2-form with respect to the natural $SU(2)$-action in $\Lambda^2(M)$. The hyperholomorphic connections are always Yang-Mills (Proposition 2.13). The Levi–Civita connection in the tangent bundle TM is a prime example of a hyperholomorphic connection (Example 2.9).

Proposition 5.1: Let M be a hyperkähler manifold, not necessary compact, and $X \subset M$ a trianalytic submanifold. Consider the normal bundle NX, equipped with a connection ∇ induced from the Levi–Civita connection on M. Then NX is hyperholomorphic.

Proof: Let L be an induced complex structure M. Consider the manifold (X, L) as a complex submanifold of (M, L). The normal bundle $NX = N(X, L)$ has a natural holomorphic structure, which is compatible with the connection ∇. Therefore, the curvature $\Theta \in \Lambda^2(X, End(NX))$ is of type $(1, 1)$ with respect to the Hodge decomposition defined by L. Thus, Θ is of type $(1, 1)$ with respect to any of the induced complex structures. By definition, this means that ∇ is hyperholomorphic. ∎

5.2 Hyperkähler embeddings are totally geodesic

In assumptions of Proposition 5.1, consider the bundle $TM\Big|_X$ with metrics and connection induced from TM. There is a natural exact sequence of holomorphic vector bundles over (X, L):

$$0 \longrightarrow TX \longrightarrow TM\Big|_X \longrightarrow NX \longrightarrow 0.$$

The natural connection in each of these bundles is hyperholomorphic (Proposition 5.1).

Proposition 5.2: Let X be a hyperkähler manifold, not necessary compact, L induced complex structure, and

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \tag{5.1}$$

be an exact sequence of holomorphic vector bundles. Let g_2 be a Hermitian structure on E_2. Consider the metrics g_1, g_2, g_3 and the connections

∇_1, ∇_2, ∇_3 on E_1, E_2, E_3 induced by g_2. Assume that ∇_1, ∇_2, ∇_3 are hyperholomorphic. Then the exact sequence (5.1) splits, and moreover, the orthogonal decomposition $E_2 = E_1 \oplus E_1^\perp$ is preserved by the connection ∇_2.

Proof: The same statement is well known for the Yang–Mills connections: every exact sequence of holomorphic bundles with compatible Yang–Mills metrics splits (see Appendix to this section). By Proposition 2.13, hyperholomorphic connections are always Yang–Mills. ∎

Definition 5.3: Let $N \stackrel{i}{\hookrightarrow} M$ be an embedding of hyperkähler manifolds. We say that i is a **hyperkähler embedding** if i is compatible with the quaternionic structure and Riemannian metric.

Corollary 5.4: Let $N \stackrel{i}{\hookrightarrow} M$ be a hyperkähler embedding. Then i is totally geodesic.

Proof: Follows directly from Proposition 4.1 and Proposition 5.2 ∎

5.3 Appendix: every exact sequence of Yang–Mills bundles splits.

Proposition 5.5: Let X be a Kähler manifold, not necessary compact, and

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0 \tag{5.2}$$

be an exact sequence of holomorphic vector bundles. Let g_2 be a Hermitian structure on E_2. Consider the induced metrics g_1, g_2, g_3. Assume that either g_1 or g_3 is Yang–Mills. Then the exact sequence (5.1) splits, and moreover, the orthogonal complement $E_1^\perp \subset E_2$ is preserved by the connection ∇_2.

Proof: Consider the second fundamental form

$$A \in \Lambda^{0,1}(X, \operatorname{Hom}(E_1, E_3))$$

of the exact sequence (5.2). The curvatures Θ_i of E_i are expressed through A as follows ([GH]):

$$\Theta_1 = \Theta_2\Big|_{E_1} + {}^t\overline{A} \wedge A, \tag{5.3}$$

$$\Theta_3 = \Theta_2\Big|_{E_3} - A \wedge {}^t\overline{A}. \tag{5.4}$$

The 2-forms ${}^t\overline{A}\wedge A$, $A\wedge {}^t\overline{A}$ are **positive** unless $A=0$. Thus, the endomorphisms $\Lambda\left(A\wedge {}^t\overline{A}\right)\in End(E_1)$, $\Lambda\left({}^t\overline{A}\wedge A\right)\in End(E_3)$ have positive trace (again, unless $A=0$). By our assumption, $\Lambda(\Theta_2)=0$ and either $\Lambda(\Theta_1)=0$ or $\Lambda(\Theta_3)=0$. Applying Λ to both sides of (5.3) and (5.4), we obtain that either $\Lambda\left(A\wedge {}^t\overline{A}\right)=0$ or $\Lambda\left({}^t\overline{A}\wedge A\right)=0$. Therefore, $A=0$ and the exact sequence (5.2) splits. ∎

6 Triholomorphic sections of hyperholomorphic bundles and deformations of trianalytic submanifolds.

In the previous section, we proved that hyperkähler embeddings are totally geodesic. In this section, we show that, furthermore, the deformational results of Section 4 are fully applicable to the deformations of trianalytic submanifolds.

6.1 Triholomorphic sections of normal bundle

Definition 6.1: Let M be a hyperkähler manifold, B a vector bundle equipped with a hyperholomorphic connection, and α a section of B. Then α is called **triholomorphic** if for each induced complex structure L on M, α is a holomorphic section of B considered as a holomorphic bundle over (M,L).

Let M be a hyperkähler manifold, not necessarily compact, and $X\subset M$ a trianalytic submanifold, not necessarily closed. Fix an induced complex structure I on M. Let

$$\left(\pi:\ \mathcal{X}\longrightarrow S, \varphi:\ \mathcal{X}\longrightarrow M, \varphi(\pi^{-1}(s_0))=X\right)$$

be a family of deformations of $(X,I)\subset(M,I)$ (see Section 4 for details). Let $\gamma:\ [0,1]\longrightarrow S$ be a real analytic path in S, such that $\gamma(0)=s_0$. Assume that for all $t\in[0,1]$, the submanifold $X_t=\varphi(\gamma^{-1}(t))\subset M$ is trianalytic. Consider the normal bundle NX to X, with the metric and connection induced from M. By Proposition 5.1, NX is hyperholomorphic. Let η be the section of NX corresponding to $\frac{d\gamma}{dt}$ as in Subsection 4.3.

Proposition 6.2: In the above assumptions, η is a triholomorphic section of NX.

Proof: Let L be an induced complex structure, and (M, L), (X, L) the manifolds M and X considered as complex manifolds with the complex structure L. The normal bundle $N(X, L)$ is naturally identified with NX as a real vector bundle. Therefore η can be considered as a section of $N(X, L)$. By Proposition 6.5 (see Appendix to this section), $\mathcal{X}$ can be considered as a real analytic deformation of (X, L). From Lemma 4.5 it is clear that η is holomorphic as a section of $N(X, L)$. This proves Proposition 6.2. ∎

6.2 Triholomorphic sections are parallel

Proposition 6.3: Let M be a hyperkähler manifold, not necessary compact, and B a vector bundle with a hyperholomorphic connection ∇. Let ν be a trianalytic section of B. Then, ν is parallel:

$$\nabla\nu = 0 \tag{6.1}$$

Proof: Let L be an induced complex structure. Since ν is triholomorphic, $\overline{\partial}_L\nu = 0$, where $\overline{\partial}_L : B \longrightarrow B \times \Lambda^{0,1}_L(M)$ is the $(0,1)$-part of the connection, taken with respect to L. Taking $L = I, -I$, we obtain

$$\overline{\partial}_I + \overline{\partial}_{-I}(\nu) = 0. \tag{6.2}$$

On the other hand, $\overline{\partial}_{-I} = \partial_I$, where ∂_I is the $(1,0)$-part of ∇ taken with respect to I. Thus,

$$\overline{\partial}_I + \overline{\partial}_{-I} = \overline{\partial}_I + \partial_I = \nabla.$$

From (6.2) we obtain that $\nabla(\nu) = 0$. ∎

From Proposition 6.2 and Proposition 6.3, we obtain that the section $\eta \in NX$ is parallel with respect to the connection. Then, Proposition 4.8 (i) can be applied to the following effect.

Corollary 6.4: Let M be a hyperkähler manifold, not necessarily compact, and $X \subset M$ a trianalytic submanifold, not necessarily closed, and

$$\big(\pi : \mathcal{X} \longrightarrow S, \varphi : \mathcal{X} \longrightarrow M, \varphi(\pi^{-1}(s_0)) = X\big)$$

be a real analytic family of deformations of $X \subset M$ (see Section 4 for details). Let $\gamma : [0,1] \longrightarrow S$ be a real analytic path in S, such that $\gamma(0) = s_0$. Assume that for all $t \in [0,1]$, the submanifolds $X_t = \varphi(\gamma^{-1}(t)) \subset M$ are trianalytic.

Fix an induced complex structure I on M. Let $U_t \subset (X_t, I)$, $t \in [0,1]$ be the subsets constructed in Subsection 4.4[1] with the corresponding holomorphic isomorphisms $\Psi_t : U_0 \longrightarrow U_t$. Then, assumptions of Proposition 4.8 (i) hold. Thus, for all $t, t' \in [0,1]$, the maps $\Psi^t_{t'}$ are isometries. Moreover, $\Psi^t_{t'}$ are compatible with the hyperkähler structure.

Proof: Assumptions of Proposition 4.8 (i) hold by Proposition 6.2 and Proposition 6.3. The maps $\Psi^t_{t'}$ are isometries by Proposition 4.8 (i). The maps $\Psi^t_{t'}$ are compatible with the hyperkähler structure because the they are obtained by integrating a certain connection in the family $\mathcal{X}$. This connection is constructed from the Hermitian metric, and thus, does not depend from the choice of induced complex structure. Taking different induced complex structures, we obtain the same maps $\Psi^t_{t'}$. Thus, $\Psi^t_{t'}$ is holomorphic with respect to each of induced complex structures (see also Lemma 4.5). ∎

6.3 Appendix: real analytic structures on hyperkähler manifolds

Consider the real analytic structures on a given hyperkähler manifold arising from the different induced complex structures. We prove that these real analytic structures are equivalent.

Proposition 6.5: Let M be a hyperkähler manifold, I_1, I_2 induced complex structures. Let (M, I_1) and (M, I_2) be the corresponding complex manifolds, and $(M, I_1)_{\mathbb{R}}$, $(M, I_2)_{\mathbb{R}}$ be the real analytic manifolds underlying (M, I_1), (M, I_2). Consider the tautological map $(M, I_1)_{\mathbb{R}} \xrightarrow{\varphi} (M, I_2)_{\mathbb{R}}$. Then φ is compatible with the real analytic structure.

Proof: Consider the **twistor space** for M (see [Bes]), $\operatorname{Tw}(M)$, with the natural holomorphic map $\pi : \operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$. Let $Sec(M)$ be the space of holomorphic sections of the map π. Then $Sec(M)$ is identified naturally with an open subspace of a Douady space for $\operatorname{Tw}(M)$, and thus, has a natural complex structure. According to [HKLR] (see [KV] for details), $Sec(M)$ is a complexification of (M, I), in the sense of Grauert. In other words, the complex valued real analytic functions on (M, I) are naturally identified with the germs of complex analytic functions on $Sec(M)$. Since $Sec(M)$ is defined independently from the choice of an induced complex structure, the tautological map φ is an equivalence. This proves Proposition 6.5. ∎

[1] By Corollary 5.4, trianalytic submanifolds are totally geodesic, and thus, Proposition 4.7 can be applied.

Proposition 6.5 implies that we may speak of a real analytic manifold underlying a given hyperkähler manifold.

Corollary 6.6: Let M be a hyperkähler manifold. Consider the Riemannian form g on M as a section of the real analytic bundle of symmetric 2-forms. Then g is real analytic.

Proof: Let I, J, K be the induced complex structures which form the standard basis in quaternions, and ω_I, ω_J, ω_K be the corresponding Kähler forms. Then $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is the natural holomorphically symplectic form on (M, I), and as such, Ω is real analytic. Then, its real part ω_J is also real analytic. Since the complex structure operator J is real analytic, we obtain that the form $g(\cdot,\cdot) := -\omega(\cdot, J\cdot)$ is also real analytic. ∎

7 Douady spaces for trianalytic cycles and real analytic structure

7.1 Real analytic structure on the Douady space.

In this section, we consider the Douady space $D_M(X)$ for a subvariety X of a compact complex manifold M equipped with a hyperkähler structure. We prove that, when X is trianalytic, $D_M(X)$ is **hypercomplex** (Definition 7.3). We also show that the real analytic variety underlying $D_M(X)$ does not change if we replace a complex structure on M by another induced complex structure. These results are technical and we use them mainly to simplify the exposition.

Let M be a compact hyperkähler manifold, and $X \subset M$ a closed trianalytic subvariety. For each induced complex structure L, consider (X, L) as a complex subvariety of (M, L). Let $D_L(X)$ be the Douady deformation space of (X, L) in (M, L). The points $[X']$ of $D_L(X)$ correspond to the subvarieties $X' \subset (M, L)$ which are deformation of X. By Theorem 2.19 (see also [V-II]), every such X' is trianalytic.

Let I_1, I_2 be induced complex structures on M. The following proposition shows that a trianalytic subvariety X' can be obtained as a deformation of (X, I_1) in (M, I_1) if and only if X' can be obtained as a deformation of (X, I_2) in (M, I_2).

Proposition 7.1: Let I_1, I_2 be induced complex structures on a compact hyperkähler manifold M, and $X \subset M$ a closed trianalytic subvariety. Consider the corresponding Douady spaces $D_{I_1}(X)$, $D_{I_2}(X)$. Then, for a trianalytic subvariety $X' \subset M$, $[X'] \in D_{I_1}(X)$ if and only if $[X'] \in D_{I_2}(X)$.

Proof: Let us recall the notion of **degree** of a trianalytic subvariety. Let M_1 be a Kähler manifold. By **degree** of a subvariety $X \subset M_1$ we understand a number

$$\deg X := \int_X \omega^{\dim_{\mathbb{C}} X},$$

where ω is the Kähler form. If $X \subset M$ is a trianalytic subvariety of a hyperkähler manifold, we can associate a number $\deg X$ to each of induced complex structures. In [V-I], we prove that $\deg X$ is in fact independent from the choice of induced complex structure. This enables us to speak of **degree** of a closed trianalytic subvariety of a compact complex manifold.

We return to the proof of Proposition 7.1. Let D_1 be the union of all components of the Douady space for (M, I_1). By [Li], [Fu], D_1 is compact; in particular, D_1 has a finite number of connected components. Let $D_1 = \cup D_1^i$, $i \in \Upsilon$ be the decomposition of D unto a union of its connected components.

Let $X' \subset M$ be a closed trianalytic subvariety of M such that $[X'] \in D_{I_2}(X)$. There exists a real analytic path $\gamma : [0,1] \longrightarrow D_{I_2}(X)$ joining $[X]$ and $[X']$. For all $t \in [0,1]$, the point $\gamma(t)$ lies in D_1^i for some $i \in \Upsilon$. Let $S^i \subset [0,1]$ be the set of all $t \in [0,1]$ such that $\gamma(t) \in D_1^i$. We are going to show that S^i are compact for all i. This will clearly imply that for all i except one, S^i is empty, thus proving that $[X]$ and $[X']$ lie in the same component of D_1.

Let $D_{\mathbb{R}}$ be the set of all real analytic deformations of X in M with natural topology. The forgetful map $\psi : D_1 \longrightarrow D_{\mathbb{R}}$ is continous and injective. Since D_1 is compact, ψ is a closed embedding. The composition $\gamma \circ \psi : [0,1] \longrightarrow D_{\mathbb{R}}$ is obviously continous, and thus, $\gamma : [0,1] \longrightarrow D_1$ is also continous. This implies that all S_i, $i \in \Upsilon$ are closed, thus proving that all S_i except one are empty. Proposition 7.1 is proven. ∎

Let $D_{I_1}(X)_{\mathbb{R}}$, $D_{I_2}(X)_{\mathbb{R}}$ be the real analytic varieties underlying $D_{I_1}(X)$, $D_{I_2}(X)$. Proposition 7.1 gives a tautological bijection

$$\psi : D_{I_1}(X)_{\mathbb{R}} \longrightarrow D_{I_2}(X)_{\mathbb{R}}.$$

Proposition 7.2: The map $\psi : D_{I_1}(X)_{\mathbb{R}} \longrightarrow D_{I_2}(X)_{\mathbb{R}}$ is an isomorphism of real analytic varieties.

Proof: Let I be an induced complex structure on M. For a real analytic function f on M, consider the function $\hat{f}: D_I(X) \longrightarrow \mathbb{R}$,

$$[X'] \xrightarrow{\hat{f}} \int_{X'} f \cdot \operatorname{Vol} X',$$

with $\operatorname{Vol} X'$ the volume form on X'. For all f, the function $\hat{f}$ is real analytic. From the definition of Douady spaces it might be seen that converging power series of different $\hat{f}$ generate the sheaf $\mathcal{A}_I$ of real analytic functions on $D_I(X)$. For an induced complex structure I', the same set $\{\hat{f}\}$ generates the sheaf $\mathcal{A}_{I'}$ of real analytic functions on $D_{I'}(X)$ (Proposition 6.5). Thus, the sheaves $\mathcal{A}_I$ and $\mathcal{A}_{I'}$ coinside. ∎

Proposition 7.2 shows that we may speak of the real analytic variety underlying $D_M(X)$ without specifying an induced complex structure I.

Definition 7.3: Let Y be a real analytic variety equipped with three complex structures I, J and K. Assume that for every point $y \in Y$, the action of I, J, K on the Zariski tangent space T_yY satisfies $I \circ J = -J \circ I = K$. Then Y is called **a hypercomplex variety**.

Remark 7.4: Proposition 7.2 immediately implies that the induced complex structures on M equip the Douady space $D_M(X)$ with a hypercomplex structure.

7.2 Appendix: isometric embeddings are hyperkähler.

Let M be a compact hyperkähler manifold, $X \subset M$ a closed trianalytic subvariety. There is an alternative way to describe the Douady space $D(X)$. The Proposition 7.5 below describes $D(X)$ in terms of of the space of isometric embeddings $\nu: X \longrightarrow M$.

Let M be a compact Kähler manifold, and $X \subset M$ be a closed analytic subvariety. One can make sense of the **fundamental class** of X, $[X] \in H^{\dim_{\mathbb{R}}(X)}(M)$, lying in homology of M. Consider X with an induced structure of a metric space, and let $X \xrightarrow{\nu} M'$ be an isometric embedding, where M' is a compact Riemannian manifold. The same argument which allows us to define the fundamental class of X, allows us to define the fundamental class of $\nu(X)$.

Proposition 7.5: Let M be a closed analytic subvariety, $X \subset M$ a closed trianalytic subvariety, and $\nu: X \longrightarrow M$ an isometric embedding. Assume that the fundamental class of $\nu(X)$ is equal to the fundamental class of X. Then $\nu(X)$ is a trianalytic subvariety of M.

Proof: We use notation introduced in Subsection 2.7. For each induced complex structure I, we have

$$\deg_I(X) - \deg_I(\nu(X)), \tag{7.1}$$

because $\nu(X)$ has the same fundamental class as X. Since X is trianalytic, applying Wirtinger's inequality (Theorem 2.26), we obtain that

$$2^n \int_{X_{ns}} \operatorname{Vol} X_{ns} = \int_{X_{ns}} \omega^n, \tag{7.2}$$

where ω is the Kähler class of (M, I), $n = \frac{1}{2}\dim_{\mathbb{R}}(X)$ and X_{ns} is the non-singular part of (X, I). By definition, we have

$$\int_{X_{ns}} \omega^n = \deg_I(X), \quad \int_{\nu(X_{ns})} \omega^n = \deg_I(\nu(X)). \tag{7.3}$$

Since ν is an isometry, and $\operatorname{Vol}(X)$ is an invariant of a metric, we have

$$\int_{X_{ns}} \operatorname{Vol} X_{ns} = \int_{\nu(X_{ns})} \operatorname{Vol}\left(\nu(X_{ns})\right). \tag{7.4}$$

Together the equations (7.1)–(7.4) give

$$2^n \int_{X_{ns}} \operatorname{Vol} X_{ns} = \int_{\nu(X_{ns})} \omega^n.$$

Applying Wirtinger's inequality (Theorem 2.26) once again, we obtain that $\nu(X)$ is complex analytic with respect to I. This proves Proposition 7.5. ∎

We just proved Theorem 1.1 (iii). As another application of Proposition 7.5, we give a direct proof that $D(X)$ is compact.

Corollary 7.6: Let M be a compact hyperkähler manifold, $X \overset{i}{\hookrightarrow} M$ be a closed trianalytic subvariety and $D(X)$ its Douady space. Then $D(X)$ is compact.

Proof: Let X_{ns} be the non-singular part of X, and $i: X_{ns} \hookrightarrow M$ the natural embedding. Then i is an isometry. Consider a deformation ν of i in class of isometries. The arument proving Proposition 7.5 shows that the closure of $\nu(X_{ns})$ is trianalytic in M. Consider the space $\underline{Is(X_{ns})}$ of isometries $X_{ns} \stackrel{\nu}{\hookrightarrow} M$. Let $Is(X_{ns})$ be a connected component of this space, containing i. There is a natural continuous surjection $p: Is(X_{ns}) \longrightarrow D(X)$, which maps $\nu \in Is(X_{ns})$ to a closure of $\nu(X_{ns})$ in M. To prove that $D(X)$ is compact it suffices to show that $Is(X_{ns})$ is compact. This is an implication of the following general statement, which is clear.

Claim 7.7: Let X, M be Riemannian manifolds, M compact. Let $Is(X, M)$ be the space of isometries (maps which preserve the geodesic distance) from X to M. Then $Is(X, M)$ is compact.

∎

8 Connections in the families of trianalytic subvarieties

8.1 Introduction

Let M be a compact hyperkähler manifold, and

$$(\pi: \mathcal{X} \longrightarrow S, \varphi: \mathcal{X} \longrightarrow M) \tag{8.1}$$

a real analytic family of subvarieties, not necessarily closed. Assume that for all $s \in S$, the fiber $X_s = \varphi(\pi^{-1}(s))$ is trianalytic in M. By Proposition 4.7 and Corollary 5.4, the family $\mathcal{X}$ is then equipped with a connection $\nabla: N_\pi X \longrightarrow T\mathcal{X}$. It is natural to ask whether this connection is **flat** (see Definition 8.4). The answer is affirmative, under certain additional assumptions.

Definition 8.1: We say that the family (8.1) **admits a compactification** if the following conditions hold.

(i) For each $s \in S$, the closure $\overline{X}_s$ of the fiber X_s in M is a trianalytic subvariety of M.

(ii) For all $s \in S$, $\overline{X}_s$ lie in the same component of the Douady space.

We show that in assumptions of Definition 8.1, the connection

$$\nabla : N_\pi X \longrightarrow T\mathcal{X}$$

is indeed flat (Theorem 8.6). However, we don't know whether it is flat for general families of trianalytic submanifolds.

8.2 Flat connections and curvature.

Definition 8.2: (curvature of a connection) Let $\pi : \mathcal{X} \longrightarrow S$ be a family of manifolds, $T_\pi X \subset T\mathcal{X}$ be the fiberwise tangent bundle and $N_\pi X = T\mathcal{X}/T_\pi X$ be a fiberwise normal bundle. Let $\nabla : N_\pi X \longrightarrow T\mathcal{X}$ be a connection in $\mathcal{X}$. Then the **curvature** of ∇ is the following tensor $\Theta \in \operatorname{Hom}_{\mathbb{R}}\left(\Lambda^2_{\mathbb{R}}\left(N_\pi X\right), T_\pi \mathcal{X}\right)$. For two sections $a, b \in N_\pi X$, consider the corresponding vector fields ∇a, $\nabla b \in T\mathcal{X}$. Consider the projection $T\mathcal{X} \xrightarrow{pr} T\mathcal{X}/\nabla N_\pi X = T_\pi X$. Let $\Theta(a,b) := pr([\nabla a, \nabla b])$. Clearly, Θ is a tensor.

Proposition 8.3: Let $(\pi : \mathcal{X} \longrightarrow S)$ be a family of manifolds equipped with a connection ∇. Then the following conditions are equivalent.

(i) The connection ∇ is flat.

(ii) Let $s_1, s_2 \in S$ be a pair of points, and $\gamma, \gamma' : [0,1] \longrightarrow S$ real analytic paths. Let $U_0 \subset X_{s_0}$, $U_1 \subset X_{s_1}$ be the open subsets such that the connection ∇ might be integrated to a map $\Psi : U_{s_0} \longrightarrow U_{s_1}$ along γ, and $U'_0 \subset X_{s_0}$, $U'_1 \subset X_{s_1}$ be such that ∇ might be integrated to a map $\Psi' : U'_{s_0} \longrightarrow U'_{s_1}$ along γ'. Then Ψ coinsides with Ψ' in the intersection $U_{s_0} \cap U'_{s_0}$.

Proof: Well known. ■

Definition 8.4: Let $\mathcal{X} \longrightarrow S$ be a family of manifolds, equipped with a connection ∇. Then ∇ is called **flat** if its curvature is zero.

8.3 Curvature of holomorphic connections

Let $\pi : \mathcal{X} \longrightarrow S$ be a complex analytic family of manifolds. Then the vector bundles $T\mathcal{X}$ and $N_\pi X$ are equipped with a natural holomorphic structure. The connection $\nabla : N_\pi X \longrightarrow T\mathcal{X}$ is called **holomorphic** if the map $\nabla : N_\pi X \longrightarrow T\mathcal{X}$ is holomorphic.

Claim 8.5: Let $\pi: \mathcal{X} \longrightarrow S$ be a family of manifolds equipped with with a holomorphic connection ∇. Consider the curvature

$$\Theta \in \operatorname{Hom}_{\mathbb{R}}\left(\Lambda^2_{\mathbb{R}}\left(N_\pi X\right), T_\pi \mathcal{X}\right).$$

Then Θ is $\mathbb{C}$-linear; in other words, Θ belongs to the subspace

$$\operatorname{Hom}_{\mathbb{C}}\left(\Lambda^2_{\mathbb{C}}\left(N_\pi X\right), T_\pi \mathcal{X}\right) \subset \operatorname{Hom}_{\mathbb{R}}\left(\Lambda^2_{\mathbb{R}}\left(N_\pi X\right), T_\pi \mathcal{X}\right).$$

Proof: Clear. ∎

8.4 Flat connections in a family of trianalytic varieties.

Let M be a compact hyperkähler manifold, $X \subset M$ a closed trianalytic subvariety, $D(X)$ its Douady space and

$$\left(\underline{\pi}: \underline{\mathcal{X}} \longrightarrow D(X), \underline{\varphi}: \underline{\mathcal{X}} \longrightarrow M\right)$$

a universal family of subvarieties corresponding to $D(X)$. Let $\mathcal{X}$ be the union of all smooth points in all fibers of $\underline{\pi}$, and π, φ be the restrictions of $\underline{\pi}$ and $\underline{\varphi}$ to $\mathcal{X}$. Consider the corresponding family of manifolds

$$\left(\pi: \mathcal{X} \longrightarrow D(X), \varphi\mathcal{X} \longrightarrow M\right).$$

We are in the same situation as described in Subsection 8.1. Thus, the family $\mathcal{X}$ is equipped with a natural connection ∇.

Theorem 8.6: In these assumptions, ∇ is flat.

Proof: The proof of Theorem 8.6 takes the rest of this section.

Clearly, $N_\pi X = \pi^* TS$. By Remark 7.4, TS has a natural quaternionic action, and thus, an action of $SU(2) \subset \mathbb{H}^*$. The bundle $T_\pi X$ is also equipped with a natural action of quaternions. This endows the bundle $\operatorname{Hom}_{\mathbb{R}}\left(\Lambda^2_{\mathbb{R}}(N_\pi X), T_\pi \mathcal{X}\right)$ with an action of $SU(2)$.

Lemma 8.7: Let Θ be a curvature of ∇, $\Theta \in \operatorname{Hom}_{\mathbb{R}}\left(\Lambda^2_{\mathbb{R}}\left(N_\pi X\right), T_\pi \mathcal{X}\right)$. Then Θ is $SU(2)$-invariant.

Proof: Let $U(1) \stackrel{\tilde{i}}{\hookrightarrow} SU(2)$ be an embedding corresponding to an algebra embedding $\mathbb{C} \stackrel{i}{\hookrightarrow} \mathbb{H}$. Let I be an induced complex structure associated with i.

The connection ∇ is holomorphic with respect to I. Therefore Θ is $\mathbb{C}$-linear with respect to the action of $\mathbb{C}$ given by $\mathbb{C} \overset{i}{\hookrightarrow} \mathbb{H}$ (Claim 8.5). Thus, Θ is $U(1)$-invariant with respect to the action of $U(1)$ given by $\widetilde{i}: U(1) \hookrightarrow SU(2)$. The group $SU(2)$ is generated by the images of $\widetilde{i}$ for all algebra embeddings $i: \mathbb{C} \hookrightarrow \mathbb{H}$. Thus, Θ is $SU(2)$-invariant. ∎

Consider $N_\pi X$, $T_\pi X$ as representation of $SU(2)$. These representations are of weight 1. The bundle

$$\operatorname{Hom}_{\mathbb{R}}\left(\Lambda^2_{\mathbb{R}}\left(N_\pi X\right), T_\pi \mathcal{X}\right)$$

is again a representation of $SU(2)$. Since $N_\pi X$, $T_\pi X$ are of weight one, $\operatorname{Hom}_{\mathbb{R}}\left(\Lambda^2_{\mathbb{R}}\left(N_\pi X\right), T_\pi \mathcal{X}\right)$ is a direct sum of representations of weight 3 and 1. Thus, this bundle cannot have non-zero $SU(2)$-invariant sections. Therefore, (Θ is $SU(2)$-invariant) implies ($\Theta = 0$). Theorem 8.6 is proven. ∎

9 Isometries of trianalytic subvarieties of a compact hyperkähler manifold

9.1 Premises

Let M be a compact hyperkähler manifold, $X \subset M$ a closed trianalytic subvariety. From Proposition 4.8, Theorem 8.6 and Corollary 6.4, we immediately obtain the following theorem.

Theorem 9.1: Let M be a compact hyperkähler manifold, $X \subset M$ a closed trianalytic subvariety, $D(X)$ the corresponding Douady space (Remark 7.4). Let $\gamma: [0,1] \longrightarrow D(X)$ be a real analytic path. Denote by X_t the subvarieties corresponding to points $\gamma(t) \in D(X)$. Consider X_t as metric spaces with the metric induced from M. Then, for each $t_1, t_2 \in [0,1]$, there exist a natural isometry $\Psi^{t_1}_{t_2}: X_{t_1} \longrightarrow X_{t_2}$, mapping non-singular points to non-singular points, and acting compatible with the hyperkähler structure on $X^{ns}_{t_1}$.[1] This isometry depends only on the homotopy class of γ.
∎

This almost finishes the proof Theorem 1.1 (i). It remains to prove the following theorem.

[1]The non-singular part X^{ns}_t is naturally equipped with a structure of a hyperkähler manifold.

Theorem 9.2: In assumptions of Theorem 9.1, let I be an induced complex structure on M. Then the isometry $\Psi_{t_2}^{t_1}: X_{t_1} \longrightarrow X_{t_2}$ is compatible with the complex analytic structure induced by I.

This section is taken fully with the proof of Theorem 9.2

9.2 Homeomorphisms of complex varieties and normalization.

Claim 9.3: ([GR]) Let X be a normal complex analytic variety, U a dense open subset in X and $f: U \longrightarrow \mathbb{C}$ a bounded holomorphic function. Then f can be extended to a holomorphic function on X.
■

This statement has an immediate corollary.

Corollary 9.4: Let $\varphi: X \longrightarrow Y$ be a continous map of complex analytic varieties. Assume that φ is holomorphic in an open dense subset $U \subset X$. Then, if X is normal, φ is holomorphic.

Proof: Let $x \in X$ be an aritrary point, V its neighbourhood, sufficiently small. Taking coordinates in a neighbourhood W of $\varphi(V)$ and applying Claim 9.3, we find that $\varphi\Big|_{U\cap V}$ extends continously to a holomorphic map $\widetilde{\varphi}$ from $U \cap V$ to W. Since $U \cap V$ is dense in V, $\widetilde{\varphi}$ coinsides with $\varphi\Big|_V$. Thus, φ is holomorphic. ■

Thus, in the situation of Theorem 9.2, were X_t normal, $\Psi_{t'}^t$ would have been holomorphic and Theorem 9.2 would have been proven. Unfortunately, we have no means to show that X_t is normal. However, from Corollary 9.4 we obtain some information about maps of arbitrary varieties too.

Corollary 9.5: Let $\varphi: X \longrightarrow Y$ be a continous map of complex analytic varieties, which is holomorphic on an open dense subset $U \subset X$. Then φ is meromorphic.

Proof: Take a normalization $\widetilde{X} \xrightarrow{n} X$. Applying Corollary 9.4 to the composition $n \circ \varphi: \widetilde{X} \longrightarrow Y$, we obtain that $n \circ \varphi$ is holomorphic. Thus, φ

is meromorphic as a composition of holomorphic $n \circ \varphi$ and a meromorphic map n^{-1}. ■

Applying Corollary 9.5 to the situation of Theorem 9.2, we obtain the following corollary.

Corollary 9.6: In the situation of Theorem 9.1, the map

$$\Psi^{t_1}_{t_2} : X_{t_1} \longrightarrow X_{t_2}$$

is bimeromorphic for each induced complex structures.

■

It remains to make a leap from "bimeromorphic" to "holomorphic". This is done in two steps. We prove a number of algebro-geometric statements about the behaviour of $\Psi^{t_1}_{t_2}$, concluding with Proposition 9.10. In Proposition 9.11, we show that these statements are strong enough to show that the $\Psi^{t_1}_{t_2}$ is holomorphic. The premise of Proposition 9.11 is purely algebro-geometric and its proof is independent from the rest of this section.

We finish this Subsection with the following statement, which we use in Proposition 9.10.

Proposition 9.7: The map $\Psi^{t_1}_{t_2} : X_{t_1} \longrightarrow X_{t_2}$ induces an isomorphism of normalizations

$$\widetilde{\Psi}^{t_1}_{t_2} : \widetilde{X}_{t_1} \longrightarrow \widetilde{X}_{t_2}.$$

Proof: From the definition of normalization [GR], the following lemma is evident.

Lemma 9.8: Let $\Psi : X \longrightarrow Y$ be a homeomorphism of complex varieties. Assume that in an open dense subset $U \subset X$, Ψ is holomorphic. Then Ψ induces an isomorphism $\widetilde{\Psi} : \widetilde{X} \longrightarrow \widetilde{Y}$ of normalizations, if the following statement holds.

(*) There exist a Stein covering $\{U_i\}$ of X such that $\{\Psi(U_i)\}$ is a Stein covering for Y.

■

To prove Proposition 9.7, it remains to show that the property **(*)** holds for the map $\Psi^{t_1}_{t_2} : X_{t_1} \longrightarrow X_{t_2}$. For each point $x \in X_{t_1}$, it suffices to construct a Stein neighbourhood U of X_{t_1}, such that $\Psi^{t_1}_{t_2}(U)$ is Stein. For a Kähler variety, consider an open ball B of radius r, taken with respect to the metric defined by geodesics. Then, for r sufficiently small, B is Stein ([GW]). Since $\Psi_{t_2}^{t_1}$ is an isometry (Theorem 9.1), an image of an open ball of radius r is again an open ball of radius r. This gives a system of Stein neighbourhoods satisfying (*). Proposition 9.7 is proven. ■

9.3 Homeomorphisms of totally geodesic subvarieties induce isomorphisms of Zariski tangent spaces.

Proposition 9.9: Let M_1, M_2 be a Kähler manifolds, $X_1^{ns} \subset M_1$, $X_2^{ns} \subset M_2$ be totally geodesic complex submanifolds, not necessarily closed, and X_1, X_2 be the closures of X_1^{ns}, X_2^{ns} in M_1, M_2. Assume that X_1, X_2 are complex analytic subvarieties of M_1, M_2. Let $\varphi: X_1 \longrightarrow X_2$ be a morphism of complex varieties, such that $\varphi\Big|_{X_1^{ns}}$ is an isometry. Then φ induces an isomorphism of Zariski tangent spaces.

Proof: To prove Proposition 9.9, we interpret the Zariski tangent space $T_x X_1$ in terms of the metric structure on X_i^{ns}, where x is a point of X_1. Let $\gamma: [0,1] \longrightarrow X_1$ be a path satisfying $\gamma([0,1]\backslash\{0.5\}) \subset X_1^{ns}$, $\gamma(0.5) = x$. There is a natural topology on the total space $Tot(TX_1^{ns})$, which comes from the embedding $Tot(TX_1) \stackrel{i}{\hookrightarrow} Tot(TM_1)$. Since X_1 is totally geodesic, i is an isometry. This topology is compatible with the map $d\varphi: TX_1 \longrightarrow TX_2$, because $d\varphi$ is also an isometry. Assume that γ is differentiable outside of $\{0.5\}$ and

$$\lim_{t\to+0.5}\frac{d\gamma}{dt} = \lim_{t\to-0.5}\frac{d\gamma}{dt}.$$

(the limits are taken in the metric completion of TX_1, which might be considered as a subset of $Tot(TM_1)$). The Zariski tangent space $T_x X_1$ can be identified with equivalence classes of such paths as follows. Two paths γ, γ' are equivalent if

$$\lim_{t\to 0.5}\frac{\rho(\gamma_1(t), \gamma_2(t))}{(t-0.5)^2} = 0,$$

where ρ is the distance function in M_1. Since the distance in M_1 coinsides with the distance in X_1, and φ is isometry, this equivalence relation is compatible with φ. ■

9.4 Algebro-geometric properties of the map $\Psi^{s_1}_{s_2}$.

Let M be a compact hyperkähler manifold, $X \subset M$ a closed trianalytic subvariety. Fix a choice of induced complex structure. Let $D(x)$ be the Douady space of X, and s_1, s_2 points on $D(X)$, and γ a real analytic path in $D(x)$ connecting s_1 and s_2. Let $X_1 = X_{s_1}, X_2 = X_{s_2}$ be the subvarieties of M corresponding to s_1, s_2, and $\Psi: \ X_1 \longrightarrow X_2$ be the bimeromorphic map of Corollary 9.6. Let X_1^{ns}, X_2^{ns} be the non-singular part of X_1, and Ψ^{ns} be the restriction of Ψ to X_1^{ns}. Let $\Gamma^{ns} \subset M \times M$ be a graph of Φ^{ns}, and Γ its closure. Clearly, Γ is a complex analytic subvariety of $M \times M$. Let $\pi_i : \ \Gamma \longrightarrow X_i$ be the projections of Γ to X_i, which are obviously morphisms of complex varieties.

Proposition 9.10: The maps π_i, $i = 1, 2$, have the following properties.

(i) π_i is finite.

(ii) π_i is dominant and induces isomorphism of normalizations

(iii) For every point $x \in \Gamma$, the differential $d\pi_i : \ T_x\Gamma \longrightarrow T_{\pi_i(x)}X_i$ is an isomorphism.

Proof: The statement (ii) follows from Proposition 9.7 and (iii) from Proposition 9.9. To prove Proposition 9.10 (i), consider the normalization map $n: \ \widetilde{\Gamma} \longrightarrow \Gamma$. On the level of rings of functions, we have an embedding

$$\mathcal{O}(X_i) \hookrightarrow \mathcal{O}(\Gamma) \hookrightarrow \mathcal{O}(\widetilde{\Gamma}).$$

Since $\widetilde{\Gamma}$ is a normalization of X_i by (ii), the ring $\mathcal{O}(\widetilde{\Gamma})$ is finitely generated as a $\mathcal{O}(X_i)$-module. Since $\mathcal{O}(X_i)$ is Noetherian ([GR]), this implies that $\mathcal{O}(\Gamma)$ is also finitely generated as a $\mathcal{O}(X_i)$-module. This proves Proposition 9.10 (i). ■

To prove that the map $\Psi: \ X_1 \longrightarrow X_2$, a.k.a. $\Psi^{s_1}_{s_2}: \ X_{s_1} \longrightarrow X_{s_2}$ is holomorphic, it suffices to show that $\pi_i : \ \Gamma \longrightarrow X_i$ is an isomorphism. In the following Subsection, we show that conditions (i) – (iii) of Proposition 9.10 are *a priori* sufficient to establish that π is an isomorphism. This will finish the proof of Theorem 9.2.

9.5 Finite dominant unramified morphisms of complex varieties.

Proposition 9.11: Let $\varphi: X \longrightarrow Y$ be a map of complex varieties satisfying conditions (i)–(iii) of Proposition 9.10. Then φ is an isomorphism.

Proof: The map φ is one-to-one in general point (by (ii)). Thus, to prove that φ is an isomorphism it suffices to show that φ is etale. On the other hand, φ is unramified by (iii). By definition of etale morphisms, to prove that φ is etale it remains to show that φ is flat. To conclude the proof of Proposition 9.11, we use the following lemma.

Lemma 9.12: Let $\varphi: X \longrightarrow Y$ be a dominant morphism of complex varieties. Assume that for every point $x \in X$, the map φ induces an isomorphism $d\varphi: T_x X \longrightarrow T_{\varphi(x)} Y$ of Zariski tangent spaces. Then φ is flat.

Proof: Let $y = \varphi(x)$. Conside the associated morphism of local rings $\mathcal{O}_y Y \stackrel{\varphi_x}{\hookrightarrow} \mathcal{O}_x X$. To prove that φ is flat, it suffices to show that φ is an isomorphism. Let $\mathfrak{m}_x$, $\mathfrak{m}_y$ be the maximal ideals in $\mathcal{O}_x X$, $\mathcal{O}_y Y$. Then $\mathfrak{m}_x/\mathfrak{m}_x^2$ is generated by $\varphi_x(\mathfrak{m}_y)$. Thus, by Nakayama, $\varphi_x(\mathfrak{m}_y)$ generate $\mathfrak{m}_x$, and we obtain $\mathfrak{m}_x = \varphi_x(\mathfrak{m}_y) \otimes \mathcal{O}_x X$. Consider $\mathcal{O}_x X$ as $\mathcal{O}_y Y$-module. Then $\mathcal{O}_x X/\mathfrak{m}_y \mathcal{O}_x X$ is one-dimensional. Applying Nakayama once more, we obtain that $\mathcal{O}_x X$ is an $\mathcal{O}_y Y$-module generated by $1 \in \mathcal{O}_x X$. Thus, the map φ_x is surjective. Since φ is dominant, φ_x is also injective. Thus, $\mathcal{O}_x X$ is a free $\mathcal{O}_{\varphi(x)} Y$-module for every $x \in X$. By the local criterion of flatness, this implies that φ is flat. Lemma 9.12 is proven. This finishes the proof of Proposition 9.11 and Theorem 9.2 ∎

10 Singular hyperkähler varieties.

It is an intriguing question, what is the "correct"[1] definition of a singular hyperkähler variety. We don't pretend to answer this question. Instead, we give an *ad hoc* set of axioms which describe some known examples (deformation spaces of stable bundles and trianalytic subvarieties). It is likely that this *ad hoc* definition is stronger than the "correct" one. A more elegant approach was suggested by Deligne and Simpson ([D], [S]).

Definition 10.1: ([V-bun], Definition 6.5) Let M be a hypercomplex variety (Definition 7.3). The following data define a structure of **hyperkähler**

[1] "Correct" in Platonic sense: some mathematicians presume that the unique "correct" definition of each and every significant mathematical object exists in itself and independently of human perception.

variety on M.

(i) For every $x \in M$, we have an $\mathbb{R}$-linear symmetric positively defined bilinear form $s_x : T_xM \times T_xM \longrightarrow \mathbb{R}$ on the corresponding real Zariski tangent space.

(ii) For each triple of induced complex structures I, J, K, such that $I \circ J = K$, we have a holomorphic differential 2-form $\Omega \in \Omega^2(M, I)$.

(iii) Fix a triple of induced complex structure I, J, K, such that $I \circ J = K$. Consider the corresponding differential 2-form Ω of (ii). Let $J : T_xM \longrightarrow T_xM$ be an endomorphism of the real Zariski tangent spaces defined by J, and $Re\Omega\Big|_x$ the real part of Ω, considered as a bilinear form on T_xM. Let r_x be a bilinear form $r_x : T_xM \times T_xM \longrightarrow \mathbb{R}$ defined by $r_x(a, b) = -Re\Omega\Big|_x (a, J(b))$. Then r_x is equal to the form s_x of (i). In particular, r_x is independent from the choice of I, J, K.

Remark 10.2:

(a) It is clear how to define a morphism of hyperkähler varieties.

(b) For M non-singular, Definition 10.1 is equivalent to the usual one (Definition 2.1). If M is non-singular, the form s_x becomes the usual Riemann form, and Ω becomes the standard holomorphically symplectic form.

(c) It is easy to check the following. Let X be a hypercomplex subvariety of a hyperkähler variety M. Then, restricting the forms s_x and Ω to X, we obtain a hyperkähler structure on X. In particular, trianalytic subvarieties of hyperkähler manifolds are always hyperkähler, in the sense of Definition 10.1.

Caution: Not everything which is seemingly hyperkähler satisfies the conditions of Definition 10.1. Take a quotient M/G os a hyperkähler manifold by an action of finite group G, acting in accordance with hyperkähler structure. Then M/G is, generally speaking, *not* hyperkähler (in fact, M/G is *never* hyperkähler). For instance, take a quotient of a 2-dimensional torus T by $G = \{\pm 1\}$ acting as an involution $t \longrightarrow -t$. This is a beautiful and well known example of a hyperkähler automorphism; the quotient space has 16 isolated singular points, which, if blown up, give a K3 surface. For x a singular point of $T/\{\pm 1\}$, its Zariski tangent space has real dimension 6. On the other hand, for a hypercomplex variety, there is a quaternion action in every Zariski tangent space, and thus, the dimension real dimension of Zariski tangent space must be divisible by 4. We obtain that the space $T/\{\pm 1\}$ is not even hypercomplex. How this happens? We take a twistor space $\operatorname{Tw}(T)$ of T and take a quotient of $\operatorname{Tw}(T)$ by the natural action of $G = \{\pm 1\}$, which is holomorphic. The quotient is a complex variety fibered over $\mathbb{C}P^1$ For $\operatorname{Tw}(T)/\{\pm 1\}$ to be hypercomplex, this fibration must be trivial, in a real analytic category. But the functor of forgetting the complex structure does not commute with taking finite quotients! Thus, even if $Tw(T)$ is (as a real analytic space) trivially fibered over $\mathbb{C}P^1$, there is no way to push down this trivialization to $\operatorname{Tw}(T)/\{\pm 1\}$.

The following theorem, proven in [V-bun] (Theorem 6.3), gives a convenient way to construct examples of hyperkähler varieties.

Theorem 10.3: Let M be a compact hyperkähler manifold, I an induced complex structure and B a stable holomorphic bundle over (M, I). Let $D(B)$ be a deformation space of stable holomorphic structures on B. Assume that $c_1(B)$, $c_2(B)$ are $SU(2)$-invariant, with respect to the standard action of $SU(2)$ on $H^*(M)$. Then $D(B)$ has a natural structure of a hyperkähler variety.

∎

The following theorem is implicit in [V-bun].

Theorem 10.4: Let M be a compact hyperkähler manifold, I an induced complex structure and $B_1, B_2, ..., B_n$ stable holomorphic bundles over (M, I). Let $D(B_i)$ be a deformation space of stable holomorphic structures on B_i. Assume that $c_1(B_i)$, $c_2(B_i)$, $i = 1, 2, ..., n$ are $SU(2)$-invariant, with respect to the standard action of $SU(2)$ on $H^*(M)$. Let Π be a natural tensor operation on the vector bundles, such that, e. g.,

$$B_1, ...B_n \longrightarrow B_1 \otimes B_2 \otimes \Lambda^2 B_3 \otimes S^7 B_4 \otimes ... \otimes B_n^*.$$

Assume that $\Pi(B_1, ...B_n)$ cannot be decomposed to a direct sum of holomorphic bundles. Then $\Pi(B_1, ...B_n)$ is stable, and the associated map

$$D(B_1) \times D(B_2) \times ..., \times D(B_n) \longrightarrow D\left(\Pi(B_1, ..., B_n)\right)$$

(defined in a certain neighbourhood of $[B_1] \times [B_2] \times ..., \times [B_n] \in D(B_1) \times D(B_2) \times ..., \times D(B_n)$ is a morphism of hyperkähler varieties.

■

Theorem 10.4 gives a natural way to construct trianalytic subvarieites of hyperkähler varieties.

The following theorem is almost trivial; the reader is advised to invent his or her own proof instead of reading ours (which is by necessity sketchy).

Theorem 10.5: Let M be a compact hyperkähler manifold, $X \subset M$ a trianalytic subvariety and $D(X)$ its Douady space. Then $D(X)$ is a compact hyperkähler variety.

Proof: Clearly, $D(X)$ is hypercomplex, in a natural way (Remark 7.4). It remains to construct the forms s_x and Ω. Let Y be a trianalytic subvariety of M which is a deformation of X, and U be a sufficiently small neighbourhood of $[Y] \in D(X)$. Let $(\pi : \mathcal{X}_U \longrightarrow U, \varphi : \mathcal{X}_U \longrightarrow M)$ be the universal family of subvarieties of M, attached to $U \subset D(X)$. The space $\mathcal{X}_U$ is hypercomplex and the map π is compatible with the hypercomplex structure. Moreover, the map $\varphi : \mathcal{X}_U \longrightarrow M$ is an immersion, so $\mathcal{X}_U$ is hyperkähler (hyperkähler structure is obtained as a pullback from M). Theorem 8.6 provides a natural trivialization of $\mathcal{X}_U \longrightarrow U$. Thus, for each $y \in Y$, there exists a natural section $\sigma_y : U \hookrightarrow \mathcal{X}_U$. By Theorem 9.2, this section is compatible with the hypercomplex structure. Restricting the hyperkähler structure from $\mathcal{X}_U$ to $\sigma_y(U)$, we obtain a hyperkähler structure on U. It is easy to check that this hyperkähler structure is independent from the choice of section σ_y. Gluing the hyperkähler structures from different U, we obtain the proof of Theorem 10.5. ■

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