Non-Hermitian Yang-Mills connections.

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Abstract. We study Yang-Mills connections on holomorphic bundles over complex Kähler manifolds of arbitrary dimension, in the spirit of Hitchin's and Simpson's study of flat connections. The space of non-Hermitian Yang-Mills (NHYM) connections has dimension twice the space of Hermitian Yang-Mills connections, and is locally isomorphic to the complexification of the space of Hermitian Yang-Mills connections (which is, by Uhlenbeck and Yau, the same as the space of stable bundles). Further, we study the NHYM connections over hyperkähler manifolds. We construct direct and inverse twistor transform from NHYM bundles on a hyperkähler manifold to holomorphic bundles over its twistor space. We study the stability and the modular properties of holomorphic bundles over twistor spaces, and prove that work of Li and Yau, giving the notion of stability for bundles over non-Kähler manifolds, can be applied to the twistors. We identify locally the following two spaces: the space of stable holomorphic bundles on a twistor space of a hyperkähler manifold and the space of rational curves in the twistor space of the "Mukai dual" hyperkähler manifold.

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1 Introduction.

1.1 An overview

In this paper we study non-Hermitian Yang-Mills (NHYM) connections on a complex vector bundle $\mathcal{B}$ over a Kähler manifold. By definition, a connection ∇ in $\mathcal{B}$ is Yang-Mills if its curvature Θ satisfies

$$\begin{cases} \Lambda(\Theta) & = \text{const} \cdot \text{id} \\ \Theta & \in \Lambda^{1,1}(M, \mathcal{E}nd(\mathcal{B})), \end{cases} \tag{1.1}$$

where Λ is the standard Hodge operator, and $\Lambda^{1,1}(M, \mathcal{E}nd(\mathcal{B})$ is the space of $(1,1)$-forms with coefficients in $\mathcal{E}nd(\mathcal{B})$ (see Definition 2.1 for details). This definition is standard [UY], [Don]. However, usually ∇ is assumed to be compatible with some Hermitian metric in $\mathcal{B}$. This is why we use the term "non-Hermitian Yang-Mills" to denote Yang-Mills connections which are not necessarily Hermitian.

An important analogy for our construction is the one with flat connections on a complex vector bundle $\mathcal{B}$. Recall that when $c_1(\mathcal{B}) = c_2(\mathcal{B}) = 0$, Hermitian Yang-Mills connections are flat (Lübcke's principle; see [Sim1]). The moduli of flat, but not necessary unitary bundles is a beautiful subject, well studied in literature (see, e. g. [Sim2]). This space has dimension twice the dimension of the moduli space of unitary flat bundles and has a natural holomorphic symplectic form. Also, generic part of the moduli of non-unitary flat bundles is equipped with a holomorphic Lagrangian[2] fibration over the space of unitary flat connections.

When $c_1(\mathcal{B}) = c_2(\mathcal{B}) = 0$, the flat connections are in one-to-one correspondence with those holomorphic structures on $\mathcal{B}$ which make it a polystable[3] holomorphic bundle [UY], [Sim1]. For arbitrary bundle $\mathcal{B}$, a similar statement holds if we replace "flat unitary" by "Hermitian Yang-Mills". Thus, it is natural to weaken the flatness assumption and consider instead all Hermitian Yang-Mills connections. The non-Hermitian Yang-Mills connections that we define correspond then to connections that are flat but not necessarily unitary. The basic properties listed above for non-unitary flat bundles hold here as well. We show that the moduli space of NHYM connections has dimension twice the dimension of the moduli of Hermitian Yang-Mills connections and is naturally equipped with a holomorphic symplectic form.

[2]Lagrangian with respect to the holomorphic symplectic form.

[3]By *polystable* we will always mean "a direct sum of stable". Throughout the paper, stability is understood in the sense of Mumford-Takemoto.

As in the case of flat bundles, generic part of the moduli of NHYM connections has a holomorphic Lagrangian fibration over the space of Hermitian Yang-Mills connections.

Let us give a brief outline of the paper. Fix a compact Kähler manifold M with a complex vector bundle $\mathcal{B}$. Let $\mathcal{M}^s$ be the set of equivalence classes of NHYM connections on $\mathcal{B}$, and let $\mathcal{M}^s_0 \subset \mathcal{M}^s$ be the subset of connections admitting a compatible Hermitian metric. Both sets turn out to have natural structures of complex analytic varieties. Recall that $\mathcal{M}^s_0$ is a moduli space of stable holomorphic bundles ([UY]; see also 2.12). After giving the relevant definitions, in Section 1 we study the structure of $\mathcal{M}^s$ in the neighborhood of $\mathcal{M}^s_0$. We prove that $\dim \mathcal{M}^s = 2 \dim \mathcal{M}^s_0$ in a neighborhood of $\mathcal{M}^s_0$. Moreover, we identify the ring of germs of holomorphic functions on $\mathcal{M}^s$ near $\mathcal{M}^s_0$ with the ring of real-analytic complex-valued functions on $\mathcal{M}^s_0$. Thus an open neighborhood $U \supset \mathcal{M}^s$ is a complexification of $\mathcal{M}^s_0$ in the sense of Grauert. We also construct a holomorphic 2-form on $\mathcal{M}^s$ which is symplectic in a neighborhood of $\mathcal{M}^s_0$. This picture is completely analogous to that for the space of flat connections, studied by Hitchin, Simpson and others ([H],[Sim2]).

For concrete examples and applications of our theory, we consider the case of NHYM-connections over a hyperkähler manifold (Definition 3.1). In this case, it is natural to modify the NHYM condition. Every hyperkähler manifold is equipped with a quaternion action in its tangent space. Since the group of unitary quaternions is isomorphic to $SU(2)$, a hyperkähler manifold has the group $SU(2)$ acting on its tangent bundle. Consider the corresponding action of $SU(2)$ on the space $\Lambda^\bullet(M)$ of differential forms over a hyperkähler manifold M. Then all $SU(2)$-invariant 2-forms satisfy (1.1) (Lemma 3.4). Thus, if the curvature Θ of a bundle $(\mathcal{B}, \nabla)$ is $SU(2)$-invariant, $\mathcal{B}$ is NHYM. Converse is *a priori* non-true when $\dim_{\mathbb{R}} M > 4$: there are 2-forms satisfying (1.1) which are not $SU(2)$-invariant. However, as the discussion at the end of Section 3 shows, for bundles over compact manifolds $SU(2)$-invariance of the curvature is a good enough approximation of the NHYM property.

A connection in $\mathcal{B}$ is called **autodual** if its curvature is $SU(2)$-invariant (Definition 3.6). For $\dim_{\mathbb{R}} M = 4$, the autoduality, in the sense of our definition, is equivalent to the anti-autoduality in the sense of 4-dimensional Yang-Mills theory. Hermitian autodual bundles were studied at great length in [V-bun].[4] Most of this paper is dedicated to the study of non-Hermitian autodual bundles over compact hyperkähler manifolds.

[4] In [V-bun], the term **hyperholomorphic** was used for "Hermitian autodual".

Consider the natural action of $SU(2)$ in the cohomology of a compact hyperkähler manifold (see the beginning of Section 3). Let $\mathcal{B}$ be a bundle with the first two Chern classes $c_1(\mathcal{B})$, $c_2(\mathcal{B})$ $SU(2)$-invariant. In [V-bun] we proved that every Hermitian Yang-Mills connection in $\mathcal{B}$ is autodual. It is natural to conjecture that in such a bundle every NHYM connection is autodual. In Theorem 3.11, we prove a weaker form of this statement: namely, in a neighbourhood of the space of Hermitian Yang-Mills connections, every NHYM connection is autodual, assuming the first two Chern classes are $SU(2)$-invariant. This is done by constructing an explicit parametrization of this neighbourhood (Proposition 2.26).

Throughout the rest of this paper (starting from Section 4) we study algebro-geometrical aspect of autodual connections. Two interdependent algebro-geometric interpretations of autoduality arise. Both of these itterpretations are related to the twistor formalism, which harks back to the works of Penrose and Salamon [Sal]. Twistor contruction is explained in detail in Section 4; here we give a brief outline of this formalism.

Every hyperkähler manifold M has a whole 2-dimensional sphere of integrable complex structures, called **induced complex structures**; these complex structures correspond bijectively to $\mathbb{R}$-algebra embeddings from complex numbers to quaternions (see Definition 3.2). We identify this 2-dimensional sphere with $\mathbb{C}P^1$. Gluing all induced complex structures together with the complex structure in $\mathbb{C}P^1$, we obtain an almost complex structure on the product $M \times \mathbb{C}P^1$ (Definition 4.1). As proven by Salamon [Sal], this almost complex structure is integrable. The complex manifold obtained in this way is called **the twistor space for** M, denoted by $\operatorname{Tw}(M)$. Consider the natural projections

$$\sigma: \operatorname{Tw}(M) = M \times \mathbb{C}P^1 \longrightarrow M, \quad \pi: \operatorname{Tw}(M) = M \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1;$$

the latter map is holomorphic. The key statement to the twistor transform is the following lemma.

Lemma 1.1 *Let $(\mathcal{B}, \nabla)$ be a bundle with a connection over a hyperkähler manifold M, and*

$$(\sigma^*\mathcal{B}, \sigma^*\nabla)$$

be the pullback of $(\mathcal{B}, \nabla)$ to the twistor space. Then $(\sigma^*\mathcal{B}, \sigma^*\nabla)$ is holomorphic if and only if $(\mathcal{B}, \nabla)$ is autodual.

Proof. This is a restatement of Lemma 5.1. ∎

This gives a natural map from the space of autodual connections on M to the space of holomorphic bundles on $\operatorname{Tw}(M)$. We prove that this map is injective, and describe its image explicitly.

For every points $x \in M$, the set $\sigma^{-1}(x)$ is a complex analytic submanifold of the twistor space. The projection $\pi\big|_{\sigma^{-1}(x)} : \sigma^{-1}(x) \longrightarrow \mathbb{C}P^1$ gives a canonical identification of $\sigma^{-1}(x)$ with $\mathbb{C}P^1$. The rational curve $\sigma^{-1}(x) \subset \operatorname{Tw}(M)$ is called **a horisontal twistor line in** $\operatorname{Tw}(M)$ (Definition 4.1).

The following proposition provides an inverse to the map given by Lemma 1.1.

Proposition 1.2 *Let M be a hyperkähler manifold, $\operatorname{Tw}(M)$ its twistor space and $\mathcal{E}$ a holomorphic bundle over $\operatorname{Tw}(M)$. Then $\mathcal{E}$ comes as a pullback of an autodual bundle $(\mathcal{B}, \nabla)$ if and only if restriction of $\mathcal{E}$ to all horisontal twistor lines is trivial as a holomorphic vector bundle. Moreover, this autodual bundle is unique, up to equivalence.*

Proof. This is a restatement of Theorem 5.12. ■

We obtained an identification of the set of equivalence classes of autodual bundles with a subset of the set of equivalence classes of bundles over the twistor space. We would like to interpret this identification geometrically, as an identification of certain moduli spaces. The autodual bundles are NHYM. The set of equivalence classes of stable NHYM bundles is equipped with a natural complex structure and is finite-dimensional, as we prove in Section 2. This general construction is used to build the moduli space of autodual bundles. It remains to define the notion of stability for holomorphic bundles over the twistor space and to construct the corresponding moduli space. The usual (Mumford-Takemoto) notion of stability does not work, because twistor spaces are not Kähler.[5] We apply results of Li and Yau [LY], who define a notion of stability for bundles over complex manifolds equipped with a Hermitian metric satisfying a certain condition (see (4.9)). The twistor space $\operatorname{Tw}(M)$ is isomorphic as a smooth manifold to $\mathbb{C}P^1 \times M$ and as such is equipped with the product metric. This metric is obviously Hermitian. We check the condition of Li and Yau for twistor spaces by computing the terms of (4.9) explicitly. This enables us to speak of stable and semistable bundles over twistor spaces.

Let $\mathcal{E}$ be a holomorphic bundle over $\operatorname{Tw}(M)$ obtained as a pullback of an autodual bundle on M. We prove that $\mathcal{E}$ is semistable. This gives a

[5] Moreover, as can be easily shown, the twistor space of a compact hyperkähler manifold admits no Kähler metric.

holomorphic interpretation of the moduli of autodual bundles on M. This is the first of our algebro-geometric interpretations. The second interpretation involves significantly more geometry, but yields a more explicit moduli space.

Let M be a compact hyperkähler manifold, and $\mathcal{B}$ a complex vector bundle with first two Chern classes invariant under the natural $SU(2)$-action. Let $\widehat{M}$ be the moduli space for the Hermitian Yang-Mills connections on $\mathcal{B}$.[6] Then $\widehat{M}$ is equipped with a natural hyperkähler structure ([V-bun]). The first result of this type was obtained by Mukai [Mu] in the context of his duality between K3-surfaces; we use the term "Mukai dual" for $\widehat{M}$ in this more general situation.

Let $X \xrightarrow{\pi} \mathbb{C}P^1$, $\widehat{X} \xrightarrow{\hat{\pi}} \mathbb{C}P^1$ be the twistor spaces for M, $\widehat{M}$, equipped with the natural holomorphic projections to $\mathbb{C}P^1$. For an induced complex structure L on the hyperkähler manifold M, we denote by (M,L) the space M considered as a complex Kähler manifold with L as a complex structure. Identifying the set of induced complex structures with $\mathbb{C}P^1$, we consider L as a point in $\mathbb{C}P^1$. Then, the complex manifold (M,L) is canonically isomorphic to the pre-image $\pi^{-1}(L) \subset X$. By i_L we denote the natural embedding $(M,L) = \pi^{-1}(L) \overset{i_L}{\hookrightarrow} X$. Let B be a stable holomorphic bundle over (M,L), with the complex vector bundle $\mathcal{B}$ as underlying complex vector space. In [V-bun], we produce a canonical identification between the moduli space of such stable bundles and the space $\widehat{M}$ of autodual connections in $\mathcal{B}$. Let $\mathcal{F}_B = i_{L*}B$ be the coherent sheaf direct image of B under i_L. The moduli space of such sheaves $\mathcal{F}_B$ is naturally identified with $\widehat{X}$ (Section 7; see also [V-bun]).

Consider a holomorphic section s of the map $\hat{\pi}: \widehat{X} \longrightarrow \mathbb{C}P^1$, that is, a holomorphic embedding $s: \mathbb{C}P^1 \longrightarrow \widehat{X}$ such that $s \circ \hat{\pi} = \mathrm{id}$. The image of such embedding is called **a twistor line in** $\widehat{X}$ (Section 4).

Let $\mathcal{E}$ be a vector bundle over X such that the pullback $i_L{}^*\mathcal{E}$ is stable for all induced complex structures $L \in \mathbb{C}P^1$. Such a bundle $\mathcal{E}$ is called **fiberwise stable** (Definition 7.1). From Lemma 7.3 it follows that fiberwise stable bundles are also stable, in the sense of Li–Yau. We restrict our attention to those bundles $\mathcal{E}$ which are, as C^∞-vector bundles, isomorphic to $\sigma^*(\mathcal{B})$, where $\mathcal{B}$ is our original complex vector bundle on M.

Every fiberwise stable bundle $\mathcal{E}$ on X gives a twistor line $s_{\mathcal{E}}: \mathbb{C}P^1 \longrightarrow \widehat{X}$ in $\widehat{X}$, where $s_{\mathcal{E}}$ associates a sheaf $i_{L*}i_L{}^*\mathcal{E}$ to $L \in \mathbb{C}P^1$. Since points of $\widehat{X}$ are identified with isomorphism classes of such sheaves, the sheaf $i_{L*}i_L{}^*\mathcal{E}$ can be naturally considered as a point in $\widehat{X}$.

Clearly, the moduli $St_f(X)$ space of fiberwise stable bundles is open in

[6]Such connections are always autodual, [V-bun].

the moduli $St(X)$ of stable bundles on X. This gives a complex structure on $St_f(X)$. The set $Sec(\widehat{X})$ of twistor lines in $\widehat{X}$ is equipped with a complex structure as a subset of the Douady space of rational curves in $\widehat{X}$.[7] We constructed a holomorphic map from $St_f(X)$ to $Sec(\widehat{X})$. We prove that this map is in fact an isomorphism of complex varieties (Theorem 7.2). The direct and inverse twistor transform give a canonical identification between the moduli of autodual bundles on M and an open subset of the moduli of semi-stable bundles on X. Thus, we obtain an identification of the moduli of autodual bundles on M and the space of twistor lines in $\widehat{X}$.

We must caution the reader that in this introduction we mostly ignore the fact that all our constructions use different notions of stability; thus, all identifications are valid only locally in the subset where all the flavours of stability hold. The precise statements are given in Sections 5–7.

1.2 Contents

- Here are the contents of our article.
- The Introduction is in two parts: the first part explains the main ideas of this paper, and the second gives an overview of its content, section by section. These two parts of Introduction are independent. The introduction is also formally independent from the main part and vice versa. The reader who prefers rigorous discourse might ignore the introduction and start reading from Section 2.
- Section 2 contains the definition of NHYM (non-Hermitian Yang-Mills) connection. We give the definition of $(0,1)$-stability for NHYM connections and consider the natural forgetful map

  $$\pi:\ \mathcal{M}^s \longrightarrow \mathcal{M}^s_0 \qquad (1.2)$$

  from the space of $(0,1)$-stable NHYM-connections to the moduli space of stable holomorphic bundles. The fiber of this map is described explicitly through a power series and the Green operator (Proposition 2.26). This map is also used to show that the moduli space of $(0,1)$-stable NHYM-connections is correctly defined and finite-dimensional (Corollary 2.9).

  Uhlenbeck–Yau theorem (Theorem 2.12; see also [UY]) provides a compatible Hermitian Yang-Mils connection for every stable holomorphic

[7] Douady spaces are analogues of Chow schemes, defined in the complex-analytic (as opposed to algebraic) setting.

bundle. This gives a section $\mathcal{M}_0^s \overset{i}{\hookrightarrow} \mathcal{M}^s$ of the map (1.2). We study the structure of $\mathcal{M}^s$ in the neighbourhood of $i(\mathcal{M}_0^s)$, and prove that this neighbourhood is isomorphic to the complexification of $\mathcal{M}_0^s$ in the sense of Grauert (Proposition 2.19).

- In Section 3, we recall the definition of a hyperkähler manifold and consider NHYM bundles over a complex manifold with a hyperkähler metric. We define autodual bundles over hyperkähler manifolds (Definition 3.6) and show that all autodual bundles are NHYM (Proposition 3.9). We cite the result of [V-bun], which shows that all Hermitian Yang-Mills connections on a bundle $\mathcal{B}$ are autodual, if the first two Chern classes of $\mathcal{B}$ satisfy a certain natural assumption ($SU(2)$-invariance; see Theorem 3.10). We also prove that, for a NHYM connection ∇ sufficiently close to Hermitian, ∇ is autodual (Theorem 3.11).

- Further on, we restrict our attention to autodual connection over hyperkähler manifolds.

- Section 4 gives a number of definition and preliminary results from algebraic geometry of the twistor spaces. We define the twistor space for an arbitrary hyperkähler manifold (Definition 4.1). The twistor space is a complex manifold equipped with a holomorphic projection onto $\mathbb{C}P^1$. For most hyperkähler manifolds (including all compact ones), the twistor space does not admit a Kähler metrics. This makes it difficult to define stability for bundles over twistor spaces. We overcome this difficulty by applying results of Li and Yau ([LY]). We consider the differential form which is an imaginary part of the natural Hermitian metric on the twistor space. To apply [LY], we compute explicitly the de Rham differential of this form (Lemma 4.4).

- In Section 5 we define the direct and inverse twistor transform relating autodual bundles over a hyperkähler manifold M and holomorphic bundles over the corresponding twistor space X. There is a map from the set of isomorphism classes of autodual bundles on M to the set of isomorphism classes of holomorphic bundles on X (Lemma 5.1). We show that this map is an embedding and describe its image explicitly (Theorem 5.12).

- In Section 6, we consider holomorphic bundles on the twistor space obtained as a result of a twistor transform. We prove semistability of such bundles. Thus, twistor transform is interpreted as a map between moduli spaces.

- In Section 7 we return to the study the algebro-geometric properties of the twistor space. For a compact hyperkähler manifold M and a stable holomorphic bundle B on M, we consider the space $\widehat{M}$ of deformations of B. When the first two Chern classes of B are $SU(2)$-invariant, the space $\widehat{M}$ has a natural hyperkähler structure; this space is called then **Mukai dual** to M. Let X, $\widehat{X}$ be the twistor spaces for M and $\widehat{M}$. We interpret the space of stable bundles on X in terms of rational curves on $\widehat{X}$ (Theorem 7.2).

- In Section 8, we relate a number of conjectures and open questions from the geometry of NHYM and autodual bundles.

2 The general case.

2.1 Definition of NHYM connections

Let M be a Kähler manifold of dimension n with the real valued Kähler form ω. Consider a complex vector bundle $\mathcal{B}$ on M. Denote by $\mathcal{A}^n(\mathcal{B})$ the bundle of smooth $\mathcal{B}$-valued n-forms on $\mathcal{B}$. Let

$$\mathcal{A}^n(\mathcal{B}) = \bigoplus_{i+j=n} \mathcal{A}^{i,j}(\mathcal{B})$$

be the Hodge type decomposition. The bundle $\mathcal{E}nd\,\mathcal{B}$ of endomorphisms of $\mathcal{B}$ is also a complex vector bundle. As usual, let

$$L : \mathcal{A}^{\bullet,\bullet}(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{\bullet+1,\bullet+1}(\mathcal{E}nd\,\mathcal{B})$$

be the operator given by multiplication ω. Let

$$\Lambda : \mathcal{A}^{\bullet+1,\bullet+1}(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{\bullet,\bullet}(\mathcal{E}nd\,\mathcal{B})$$

be the adjoint operator with respect to the trace form on $\mathcal{E}nd\,\mathcal{B}$.

Definition 2.1 A connection $\nabla : \mathcal{B} \to \mathcal{A}^1(\mathcal{B})$ is called **non-Hermitian Yang-Mills** (NHYM for short) if its curvature $R \in \mathcal{A}^2(\mathcal{E}nd\,\mathcal{B})$ is of Hodge type $(1,1)$ and satisfies

$$\Lambda \circ R = c\,\mathrm{Id}$$

for a certain constant $c \in \mathbb{C}$.

Remark 2.2 This terminology is perhaps unfortunate, in that a NHYM connection can (but need not) be Hermitian. We use the term for lack of better one.

To simplify exposition, we will always consider only NHYM connections with the constant $c = 0$.

Let ∇ be a NHYM connection on $\mathcal{B}$. Since the $(0,2)$-component of its curvature vanishes, the $(0,1)$-component $\nabla^{0,1} : \mathcal{B} \to \mathcal{A}^{0,1}(\mathcal{B})$ defines a holomorphic structure on $\mathcal{B}$. We will call this **the holomorphic structure associated to** ∇.

Let $\overline{\mathcal{B}}^*$ be the dual to the complex-conjugate to the complex bundle $\mathcal{B}$. Every NHYM-connection ∇ obviously induces a NHYM connection ∇^* on the dual bundle $\mathcal{B}^*$. Let $\overline{\nabla}$ be the connection on $\overline{\mathcal{B}}^*$ complex-conjugate to ∇^*. The connection $\overline{\nabla}$ is also obviously NHYM. We will call it **the adjoint connection to** ∇. The holomorphic structure on $\overline{\mathcal{B}}^*$ associated to $\overline{\nabla}$ will be called **the adjoint holomorphic structure associated to** ∇. Note that the adjoint holomorphic structure depends only on the $(1,0)$-part $\nabla^{1,0}$ of the connection ∇.

2.2 Stability and moduli of NHYM connections.

Fix a compact Kähler manifold M and a complex vector bundle $\mathcal{B}$ on M. Consider the space $\mathcal{A}$ of all connections on $\mathcal{B}$ and let $\mathcal{A}_0$ be the subspace of NHYM connnections. The space $\mathcal{A}$ is a complex-analytic Banach manifold, and $\mathcal{A}_0 \subset \mathcal{A}$ is an analytic subspace of $\mathcal{A}$. Let $\mathcal{G} = \operatorname{Maps}(M, \operatorname{Aut}\mathcal{B})$ be the complex Banach-Lie group of automorphisms of $\mathcal{B}$. The group $\mathcal{G}$ acts on $\mathcal{A}$ preserving the subset $\mathcal{A}_0$.

In order to obtain a good moduli space for NHYM connections, we need to impose some stability conditions.

Definition 2.3 A NHYM connection ∇ is called $(0,1)$**-stable** if the bundle $\mathcal{B}$ with the associated holomorphic structure is a stable holomorphic bundle.

Further on, we sometimes use the term *stable* to denote $(0,1)$-stable connections.

Remark 2.4 This definition is sufficient for our present purposes. However, it is unnaturally restrictive. See 8.3 for a more natural definition.

Let $\mathcal{A}^s \subset \mathcal{A}_0$ be the open subset of $(0,1)$-stable NYHM connections and let

$$\mathcal{M}^s = \mathcal{A}_s/\mathcal{G}$$

be the set of *equivalence classes* of $(0,1)$-stable NHYM connections on $\mathcal{B}$ endowed with the quotient topology.

Choose a connection $\nabla \in \mathcal{A}_0, \nabla : \mathcal{B} \to \mathcal{A}^1(\mathcal{B})$. Let $\nabla = \nabla^{1,0} + \nabla^{0,1}$ be the type decomposition and extend both components to differentials

$$D : \mathcal{A}^{\bullet,0}(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{\bullet+1,0}(\mathcal{E}nd\,\mathcal{B})$$
$$\overline{D} : \mathcal{A}^{0,\bullet} \to \mathcal{A}^{0,\bullet+1}(\mathcal{E}nd\,\mathcal{B})$$

The tangent space $T_\nabla(\mathcal{A})$ equals $T_\nabla(\mathcal{A}) = \mathcal{A}^1(\mathcal{E}nd\,\mathcal{B})$. The NHYM equations define a complex-analytic map

$$YM : \mathcal{A} \to \mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^{0,2} \oplus \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B}).$$

It is easy to see that the differential of YM at the point ∇ is given by

$$YM_\nabla = D + \overline{D} + \Lambda\nabla : \mathcal{A}^1(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^{0,2}(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B})$$

On the other hand, the differential at ∇ of the $\mathcal{G}$-action on $\mathcal{A}$ is given by

$$\nabla : \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^1(\mathcal{E}nd\,\mathcal{B})$$

Definition 2.5 The complex

$$0 \to \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^1(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^{0,2}(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B})$$

is called **the deformation complex** of the NHYM connection ∇.

Remark 2.6 The deformation complex has a natural structure of a differential graded Lie algebra.

Proposition 2.7 *The deformation complex is elliptic.*

Proof. Indeed, the complex

$$0 \to \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{0,1}(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{0,2}(\mathcal{E}nd\,\mathcal{B})$$

is the Dolbeault complex for the holomorphic bundle $\mathcal{E}nd\,\mathcal{B}$ and is therefore elliptic. Hence it is enough to prove that

$$0 \to \mathcal{A}^{1,0}(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B}) \tag{2.3}$$

is elliptic. By Kodaira identity $\Lambda D = \sqrt{-1}D^*$ on $\mathcal{A}^{1,0}(\mathcal{E}nd\,\mathcal{B})$, and this complex is the same as

$$0 \to D \oplus D^* : \mathcal{A}^{1,0}(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^0(\mathcal{E}nd\,\mathcal{B}),$$

where D^* is defined by means of the trace form on $\mathcal{E}nd\,\mathcal{B}$. This complex is obviously elliptic. ∎

Corollary 2.8 *Let $\widetilde{G} \subset \mathcal{G}$ be the stabilizer of $\nabla \in \mathcal{A}$. Then*

1. *$\widetilde{G}$ is a finite dimensional complex Lie group.*
2. *There exists a finite dimensional locally closed complex-analytic Stein subspace $\widetilde{\mathcal{M}^s} \subset \mathcal{A}$ containing ∇ and invariant under $\widetilde{G}$ such that the natural projection*
$$\widetilde{\mathcal{M}^s}/\widetilde{G} \to \mathcal{M}^s$$
is an open embedding.

Proof. This is the standard application of the Luna's slice theorem, see [Kob]. ∎

Corollary 2.9 *The topological space $\mathcal{M}^s$ has a natural structure of a complex-analytic space.*

Proof. Indeed, since $\widetilde{\mathcal{M}^s}$ is Stein, the quotient $\widetilde{\mathcal{M}^s}/\widetilde{G}$ is a Stein complex-analytic space. Now by the standard argument ([Kob]) the induced complex analytic charts on $\mathcal{M}^s$ glue together to give a complex-analytic structure on the whole of $\mathcal{M}^s$. ∎

2.3 Hermitian Yang-Mills bundles and the theorem of Uhlenbeck–Yau

For every complex bundle $\mathcal{B}$ on M denote by $\mathcal{M}_0^s(\mathcal{B})$ the moduli space of stable holomorphic structures on $\mathcal{B}$. Fix $\mathcal{B}$ and consider the space $\mathcal{M}^s$ of $(0,1)$-stable NHYM connections on $\mathcal{B}$. Taking the associated holomorphic structure defines a map $\pi : \mathcal{M}^s \to \mathcal{M}_0^s(\mathcal{B})$.

Lemma 2.10 *The map π is holomorphic.*

Proof. Clear. ∎

Since every complex vector bundle admits an Hermitian metric, the complex vector bundles $\mathcal{B}$ and $\overline{\mathcal{B}}^*$ are isomorphic. Therefore the moduli spaces $\mathcal{M}_0^s(\mathcal{B})$ and $\mathcal{M}_0^s(\overline{\mathcal{B}}^*)$ are naturally identified. Denote the space $\mathcal{M}_0^s(\mathcal{B}) = \mathcal{M}_0^s(\overline{\mathcal{B}}^*)$ simply by $\mathcal{M}_0^s$, and let $\overline{\mathcal{M}}_0^s$ be the complex-conjugate space.

Consider the open subset $\mathcal{M}^{gs} \subset \mathcal{M}^s$ of $(0,1)$-stable NHYM connections on $\mathcal{B}$ such that the adjoint connection on $\overline{\mathcal{B}}^*$ is also $(0,1)$-stable. Taking the adjoint holomorphic structure defines a map $\overline{\pi} : \mathcal{M}^{gs} \to \overline{\mathcal{M}}_0^s$. This map is also obviously holomorphic.

Lemma 2.11 *A NHYM connection $\nabla \in \mathcal{M}^{gs}$ satisfies $\pi(\nabla) = \overline{\pi}(\nabla)$ if and only if it admits a compatible Hermitian metric.*

Proof. Indeed, $\pi(\nabla) = \overline{\pi}(\nabla)$ if and only if there exists an isomorphism $h : \mathcal{B} \to \overline{\mathcal{B}}^*$ sending ∇ to $\overline{\nabla}$. This isomorphism defines an Hermitian metric on $\mathcal{B}$ compatible with the connection ∇. ∎

To proceed further we need to recall the following fundamental theorem.

Theorem 2.12 (Uhlenbeck,Yau) *Every stable holomorphic bundle $\mathcal{B}$ on a Kähler manifold M admits a unique Hermitian Yang-Mills connection ∇. Vice versa, every holomorphic bundle admitting such a connection is polystable.*

We will call such a metric **a Uhlenbeck-Yau metric** for the holomorphic bundle $\mathcal{B}$.

Corollary 2.13 *Let $\mathcal{M}^s_u \in \mathcal{M}^{gs}$ be subset of equivalence classes of Hermitian connections. The product map $\pi \times \overline{\pi} : \mathcal{M}^{gs} \to \mathcal{M}^s_0 \times \overline{\mathcal{M}}^s_0$ identifies $\mathcal{M}^s_u$ with the diagonal in $\mathcal{M}^s_0 \times \overline{\mathcal{M}}^s_0$.*

Proof. Clear. ∎

Remark 2.14 Note that the subset $\mathcal{M}^s_u \subset \mathcal{M}^{gs}$ is not complex-analytic, but only real-analytic.

2.4 Moduli of NHYM connections as a complexification of the moduli of stable bundles

Let $\nabla \in \mathcal{M}^s_u \subset \mathcal{M}^s$ be an Hermitian Yang-Mills connection, and let $\mathcal{F}_\nabla = \pi^{-1}(\pi(\nabla)) \subset \mathcal{M}^s$ be the fiber of π over ∇. In order to study the map $\pi \times \overline{\pi} : \mathcal{M}^{gs} \to \mathcal{M}^s_0 \times \overline{\mathcal{M}}^s_0$ in a neighborhood of ∇, we first study the restriction of the map $\overline{\pi}$ to $\mathcal{F}_\nabla$. We begin with the following.

Theorem 2.15 *Let $\overline{D} : \mathcal{B} \to \mathcal{A}^{0,1}(\mathcal{B})$ be a representative in the equivalence class $\pi(\nabla) \in \mathcal{M}^s_0$ of holomorphic structures on $\mathcal{B}$ and let $D : \mathcal{B} \to \mathcal{A}^{1,0}(\mathcal{B})$ be the operator adjoint to $\overline{D}$ with respect to the Uhlenbeck-Yau metric. The fiber $\mathcal{F}_\nabla$ is isomorphic to the set of all $\mathcal{E}nd\,\mathcal{B}$-valued $(1,0)$-forms θ satisfying*

$$\begin{cases} D\theta + \theta \wedge \theta & = 0, \\ D^*\theta & = 0. \end{cases} \tag{2.4}$$

Proof. To define the desired isomorphism, choose for any equivalence class $\nabla_1 \in \pi^{-1}(\pi(\nabla))$ a representative $\nabla_1 = \nabla_1^{1,0} + \overline{D} : \mathcal{B} \to \mathcal{A}^1(\mathcal{B})$. Every two representatives must differ by a gauge transformation $g : \mathcal{B} \to \mathcal{B}$. The map g must preserve the holomorphic structure $\overline{D}$. However, this holomorphic structure is by assumption stable. Therefore $g = c\,\mathrm{Id}$ for $c \in \mathbb{C}$, and the operator $\nabla_1^{1,0} : \mathcal{B} \to \mathcal{A}^{1,0}(\mathcal{B})$ is defined uniquely by its class in $\mathcal{F}_\nabla$. Take $\theta = \nabla_1^{1,0} - D$; the equations (2.4) follow directly from the definition of NHYM connections. ∎

In order to apply this Theorem, note that by the second of the equations (2.4) every NHYM connection $\nabla_1 \in \mathcal{F}_\nabla$ defines a D^*-closed $\mathcal{E}nd\,\mathcal{B}$-valued $(1,0)$-form θ. Complex conjugation with respect to the Uhlenbeck-Yau metric h_∇ identifies the space of D^*-closed $\mathcal{E}nd\,\mathcal{B}$-valued $(1,0)$-forms with the space of $\overline{D}^*$-closed $(0,1)$-forms, and it also identifies the respective cohomology spaces. But the cohomology spaces of the Dolbeault complex $\mathcal{A}^{0,\bullet}(\mathcal{E}nd\,\mathcal{B})$ with respect to $\overline{D}$ and $\overline{D}^*$ are both equal to the space of harmonic forms, hence naturally isomorphic. Collecting all this together, we define a map $\rho : \mathcal{F}_\nabla \to \overline{H^1(M, \mathcal{E}nd\,\mathcal{B})}$ by the rule

$$\nabla_1 \mapsto \langle \text{ class of } \theta \text{ in } H^1(M, \mathcal{E}nd\,\mathcal{B})\rangle. \tag{2.5}$$

Proposition 2.16 *The map ρ is a closed embedding in a neighborhood of $\nabla \in \mathcal{M}_u^s \in \mathcal{F}_\nabla$.*

This Proposition can be deduced directly from Theorem 2.15. However, we prefer to prove a stronger statement. To formulate it, consider the adjoint holomorphic structure $\overline{\pi}(\nabla)$ on the complex bundle $\overline{\mathcal{B}}^* \cong \mathcal{B}$. Recall the following standard fact from the deformation theory of holomorphic bundles.

Theorem 2.17 *Let $\bar\partial : \mathcal{B} \to \mathcal{A}^{0,1}(\mathcal{B})$ be a stable holomorphic structure on a complex Hermitian bundle $\mathcal{B}$. There exists a neighborhood $U \subset \mathcal{M}_0^s$ of $\bar\partial$ such that every $\bar\partial_1 \in U$ can be represented uniquely by an operator $\bar\partial + \theta : \mathcal{B} \to \mathcal{A}^{0,1}(\mathcal{B})$ satisfying*

$$\begin{cases} \bar\partial\theta = \theta \wedge \theta \\ \bar\partial^*\theta = 0 \end{cases} \tag{2.6}$$

The **Kuranishi map** *$U \to H^1(M, \mathcal{E}nd\,\mathcal{B})$ defined by*

$$\bar\partial_1 \to \langle \textit{ class of } \theta \textit{ in } H^1_{\bar\partial^*}(\mathcal{A}^{0,\bullet}(\mathcal{E}nd\,\mathcal{B})) \cong H^1(M, \mathcal{E}nd\,\mathcal{B})\rangle$$

is a locally closed embedding.

Corollary 2.18 *The map $\overline{\pi} : \mathcal{F}_\nabla \to \overline{\mathcal{M}}_0^s$ is biholomophic in a neighborhood V of $\nabla \in \mathcal{M}_u^s \subset \mathcal{M}^{gs}$, and the map $\rho : V \to H^1(M, \mathcal{E}nd\,\mathcal{B})$ is the composition of $\overline{\pi} : \mathcal{F}_\nabla \to \overline{\mathcal{M}}_0^s$ and the Kuranishi map.*

Proof. Complex conjugation sends equations (2.4) precisely to (2.6), and thus establishes a bijection between neighborhoods of $\nabla \in \mathcal{F}_\nabla$ and $\overline{\pi}(\nabla) \in \overline{\mathcal{M}}_0^s$. ∎

This statement, in turn, implies the following.

Proposition 2.19 *The product map $\pi \times \overline{\pi} : \mathcal{M}^{gs} \to \mathcal{M}_0^s \to \overline{\mathcal{M}}_0^s$ is biholomorphic on an open neighborhood U of the subset $\mathcal{M}_u^s \in \mathcal{M}^{gs}$.*

Proof. Consider both $\mathcal{M}^{gs}$ and $\mathcal{M}_0^s \times \overline{\mathcal{M}}_0^s$ as spaces over $\mathcal{M}_0^s$. The map $\pi \times \overline{\pi}$ is a map over $\mathcal{M}_0^s$, and it is locally biholomorphic on every fiber of the natural projections $\mathcal{M}^{gs} \to \mathcal{M}_0^s$, $\mathcal{M}_0^s \times \overline{\mathcal{M}}_0^s \to \mathcal{M}_0^s$. ∎

Thus $\dim \mathcal{M}^s = 2 \dim \mathcal{M}_0^s$, and an open neighborhood U of the subspace $\mathcal{M}_u^s \subset \mathcal{M}^{gs}$ is the complexification of $\mathcal{M}_0^s$ is the sense of Grauert.

2.5 Holomorphic symplectic form on the moduli of NHYM bundles

In order to construct a holomorphic symplectic 2-form on $\mathcal{M}^s$, we need to restrict our attention to a smooth open subset of $\mathcal{M}^s$.

Definition 2.20 A NHYM connection ∇ is called **smooth** if both $\nabla \in \mathcal{M}^s$ and $\pi(\nabla) \in \mathcal{M}_0^s$ are smooth points.

Let $\nabla \in \mathcal{M}^s$ be a smooth NHYM connection and denote by $\mathcal{C}_\nabla$ its deformation complex. By construction the holomorphic tangent space $T_\nabla(\mathcal{M}^s)$ is identified with a subspace of the first cohomology space $H^1(\mathcal{C}_\nabla)$. By definition $\pi(\nabla)$ is a smooth point, and the tangent space $T_{\pi(\nabla)}(\mathcal{M}_0^s)$ is a subspace of the first cohomology space $H^1(M, \mathcal{E}nd\,\mathcal{B})$ of M with coefficients in $\mathcal{B}$ equipped with the induced holomorphic structure.

Consider the natural projection map $\mathcal{C}_\nabla \to \mathcal{A}^{0,i}(M, \mathcal{E}nd\,\mathcal{B})$ from $\mathcal{C}_\nabla$ to the Dolbeault complex of the bundle $\mathcal{E}nd\,\mathcal{B}$. Denote by

$$\mathrm{pr} : H^1(\mathcal{C}_\nabla) \to H^1(M, \mathcal{E}nd\,\mathcal{B})$$

the induced map on the cohomology spaces and let

$$W = \operatorname{Ker} \mathrm{pr} \subset H^1(\mathcal{C}_\nabla).$$

Proposition 2.21 *Let $d\pi : T_\nabla(\mathcal{M}^s) \to T_{\pi(\nabla)}(\mathcal{M}_0^s)$ be the differential of the map $\pi : \mathcal{M}^s \to \mathcal{M}_0^s$ in the smooth point $\nabla \in \mathcal{M}_0^s$. The diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_\nabla(\mathcal{F}_\nabla) & \longrightarrow & T_\nabla(\mathcal{M}^s) & \xrightarrow{d\pi} & T_{\pi(\nabla)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
9 & \longrightarrow & W & \longrightarrow & H^1(\mathcal{C}_\nabla) & \xrightarrow{\mathrm{pr}} & H^1(M, \mathcal{E}nd\,\mathcal{B}) & \longrightarrow & 0
\end{array}$$

is commutative.

Proof. Clear. ∎

We first construct a symplectic form on the space $H^1(\mathcal{C}_\nabla)$. To do this, we first identify the space $W \subset H^1(\mathcal{C}_\nabla)$.

Lemma 2.22 *The space W is naturally isomorphic to $H^{n-1}(M, \mathcal{E}nd\,\mathcal{B} \otimes \mathcal{K})$, where K is the canonical line bundle on M.*

Proof. Indeed, the space W is isomorphic to the space of $\mathcal{E}nd\,\mathcal{B}$-valued $(1,0)$-forms satisfying

$$\begin{cases} D\theta = 0 \\ \Lambda\overline{D}\theta = 0 \end{cases} \tag{2.7}$$

Consider the map

$$\bullet \wedge \omega^{n-1} : \mathcal{A}^{1,0}(\mathcal{E}nd\,\mathcal{B}) \to \mathcal{A}^{n,n-1}(\mathcal{E}nd\,\mathcal{B}).$$

By Kodaira identities a form $\theta \in \mathcal{A}^1(M, \mathcal{E}nd\,\mathcal{B})$ satisfies (2.7) if and only if $\theta \wedge \omega^{n-1}$ is harmonic. Hence $\bullet \wedge \omega^{n-1}$ identifies W with $H^{n-1}(M, \mathcal{E}nd\,\mathcal{B} \otimes \mathcal{K})$. ∎

Consider now two $\mathcal{E}nd\,\mathcal{B}$-valued 1-forms $\theta_0, \theta^1 \in \mathcal{A}^1(M, \mathcal{E}nd\,\mathcal{B})$ and let

$$\Omega(\theta_0, \theta_1) = \int_M \mathrm{tr}(\theta_0 \wedge \theta_1 \wedge \omega^{n-1}),$$

where tr is the trace map, $n = \dim M$ and ω is the Kähler form on M.

Lemma 2.23 *If $\theta_0 = \nabla g$ for some section $g \in \mathcal{A}^0(M, \mathcal{E}nd\,\mathcal{B})$, then*

$$\Omega(\theta_0, \theta_1) = 0$$

for any $\theta_1 \in \mathcal{A}^1(M, \mathcal{E}nd\,\mathcal{B})$ satisfying $\Lambda\nabla(\theta_1) = 0$.

Proof. Indeed,

$$\Omega(\theta_0, \theta_1) = \int_M \operatorname{tr}(\nabla(\omega^{n-1} g)) \wedge \theta_1) =$$
$$= \int_M \operatorname{tr}(\omega^{n-1} \wedge \nabla(g\theta_1)) - \int_M \operatorname{tr}(g\omega^{n-1} \wedge \nabla(\theta_1)) = \int_M \operatorname{tr}(g\Lambda\nabla(\theta_1))\omega^n = 0$$

■

Corollary 2.24 *1. The form Ω defines a complex 2-form on the space $H^1(\mathcal{C}_\nabla)$.*

2. The subspace W is isotropic, and the induced pairing

$$(W \cong H^{n-1}(M, \mathcal{E}nd\,\mathcal{B} \otimes \mathcal{K})) \otimes H^1(M, \mathcal{E}nd\,\mathcal{B}) \to \mathbb{C}$$

is non-degenerate.

Proof. The first statement is clear. It is easy to see that the pairing induced by Ω is exactly the one defined by the Serre duality, which proves the second statement. ■

Restricting to the subspace $T_\nabla(\mathcal{M}^s)$, we get a 2-form Ω on $\mathcal{M}^s$. This form is obviously holomorphic.

Proposition 2.25 *Assume that either $\nabla \in \mathcal{M}^s$ is Hermitian, or $T_\nabla(\mathcal{M}^s) = H^1(\mathcal{C}_\nabla)$. Then the form Ω on $T_\nabla(\mathcal{M}^s)$ is non-degenerate. The map $\mathcal{M}^s \to \mathcal{M}^s_0$ is a Lagrangian fibration in the neghrborhood of ∇.*

Proof. It is easy to see that $T_\nabla(\mathcal{F}_\nabla) \subset T_\nabla(\mathcal{M}^s)$ is isotropic. If the connection ∇ is *unobstructed*, that is, the embedding $T_\nabla(\mathcal{M}^s) \hookrightarrow H^1(\mathcal{C}_\nabla)$ is actually an isomorphism, then the statement follows from Corollary 2.24.

Suppose now that the inclusion $T_\nabla(\mathcal{M}^s) \subset H^1(\mathcal{C}_\nabla)$ is proper. By assumption the connection ∇ is Hermitian in this case, therefore the complex conjugation map $\overline{} : \mathcal{A}^{0,1}(M, \mathcal{E}nd\,\mathcal{B}) \to \underline{\mathcal{A}^{1,0}(\mathcal{E}nd\,\mathcal{B})}$ is defined. It is easy to see that this map identifies $\operatorname{Ker}\operatorname{pr}$ with $\overline{H^1(M, \mathcal{E}nd\,\mathcal{B})}$. By Corollary 2.18 it also identifies $T_\nabla(\mathcal{F}_\nabla)$ with $\overline{T_{\pi(\nabla)}}$.

The form $\Omega(\bullet, \overline{\bullet})$ is a non-degenerate Hermitian form on $\operatorname{Ker}\operatorname{pr}$. Therefore its restriction to $T_\nabla(M_\nabla)$ is also non-degenerate.

The last statement now follows directly from Proposition 2.21 and Corollary 2.24. ■

2.6 Local parametrization of the moduli of NHYM connections

In the last part of this section we give a more explicit description of the embedding $\mathcal{F}_\nabla \to H^1(\mathcal{E}nd\,\mathcal{B})$ for an Hermitian Yang-Mills connection ∇ in the spirit of [V-bun]. This description is of independent interest, and we will also use it in the next section in the study of NHYM connections on hyperkähler manifolds.

Fix an Uhlenbeck-Yau metric on $\mathcal{B}$ compatible with ∇ and let $\Delta = DD^* + D^*D$ be the associated Laplace operator on $\mathcal{A}^\bullet(\mathcal{E}nd\,\mathcal{B})$. Let G be the Green operator provided by the Hodge theory. Recall that we have the Hodge decomposition

$$\mathcal{A}^\bullet(\mathcal{E}nd\,\mathcal{B}) = \mathcal{H}^\bullet(\mathcal{E}nd\,\mathcal{B}) \oplus \mathcal{A}^\bullet_{\mathrm{ex}}(\mathcal{E}nd\,\mathcal{B})$$

into the space of harmonic form $\mathcal{H}^\bullet$ and its orthogonal complement $\mathcal{A}^\bullet_{\mathrm{ex}}$. This complement is further decomposed as

$$\mathcal{A}^\bullet_{\mathrm{ex}}(\mathcal{E}nd\,\mathcal{B}) = D\mathcal{A}^{\bullet-1}(\mathcal{E}nd\,\mathcal{B}) \oplus D^*\mathcal{A}^{\bullet+1}(\mathcal{E}nd\,\mathcal{B})$$

The composition DD^*G is by definition the projection onto $D\mathcal{A}^{\bullet-1}(\mathcal{E}nd\,\mathcal{B}) \subset \mathcal{A}^\bullet(\mathcal{E}nd\,\mathcal{B})$.

Take now a small neighborhood $U \subset \mathcal{F}_\nabla$ of the Hermitian connection $\nabla \in \mathcal{F}_\nabla$ and a NHYM connection $\nabla_1 \in U$. Let $\theta = \nabla_1 - \nabla$, $\theta \in \mathcal{A}^{1,0}(M, \mathcal{E}nd\,\mathcal{B})$ and let $K(\theta) \in \overline{H^{1,0}(M, \mathcal{E}nd\,\mathcal{B})}$ be the associated cohomology class. Shrinking U if neccesary, we see that by Proposition 2.16 the connection ∇_1 is uniquely determined by the class $K(\theta)$.

Let $\theta_0 \in \mathcal{A}^{1,0}(M, \mathcal{E}nd\,\mathcal{B})$ be the harmonic form representing the class $K(\theta)$. Define by induction

$$\theta_n = D^*G \sum_{0\leq k<n} \theta_k \wedge \theta_{n-1-k}.$$

Proposition 2.26 *Let ∇ be an Hermitian Yang-Mills connection. There exists a neighborhood $V \subset \mathcal{F}_\nabla$ of $\nabla \in \mathcal{F}_\nabla$ such that for every $\nabla_1 \in V \subset \mathcal{F}_\nabla$ the series*

$$\sum_{0\leq k} \theta_k \tag{2.8}$$

converges to the form $\theta = \nabla_1 - \nabla$.

Proof. The metric on $\mathcal{E}nd\,\mathcal{B}$ defines a norm $\|\bullet\|$ on $\mathcal{A}^{\bullet,0}(\mathcal{E}nd\,\mathcal{B})$. We can assume that $\|\theta_0\| < \varepsilon$ for any fixed $\varepsilon > 0$. Since the Hodge decomposition is orthogonal,

$$\|D\theta_n\| = \left\| DD^*G\left(\sum_{0\le k<n} \theta_k \wedge \theta_{n-k}\right)\right\|$$
$$\le \left\|\sum_{0\le k<n} \theta_k \wedge \theta_{n-k}\right\| \le \sum_{0\le k<n} \|\theta_k\| \cdot \|\theta_{n-k}\|.$$

Since $D : D^*(\mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B})) \to \mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B})$ is injective and elliptic, there exists a constant $C > 0$ such that

$$\|Df\| > C\|f\|$$

for all $f \in D^*(\mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B}))$. Let $a_n = \frac{(2n)!}{(n!)^2}$ be the Catalan numbers. By induction

$$\|\theta_n\| < a_n \left|\frac{\varepsilon}{C}\right|.$$

Since $A(z) = \sum a_n z^n$ satisfies $A(z) = 1 + z(A(z))^2$, it equals

$$A(z) = \frac{1-\sqrt{1-4z}}{2}$$

and converges for $z < \frac{1}{4}$. Therefore the series (2.8) converges for $4\varepsilon < C$.

To prove that it converges to θ, let $\chi_0 = \theta$ and let

$$\chi_n = \theta - \sum_{0\le k<n} \theta_k$$

for $n \ge 1$. Since both ∇ and $\nabla + \theta$ are NHYM, we have $D\chi_1 = \chi_0 \wedge \chi_0$. Therefore $\chi_0 \wedge \chi_0 \in \mathcal{A}^{2,0}(\mathcal{E}nd\,\mathcal{B})$ and

$$D\chi_1 = DD^*G(\chi_0 \wedge \chi_0) = \chi_0 \wedge \chi_0.$$

By induction

$$D\chi_n = \chi_0 \wedge \chi_{n-1} + \sum_{0\le k\le n} \chi_k \wedge \theta_{n-1-k},$$

and $\chi_n \in D^*(\mathcal{A}^{2,0}(M, \mathcal{E}nd\,\mathcal{B}))$ for all $n > 0$. Again by induction,

$$\|\chi_n\| < a_{n+1}\left(\frac{\varepsilon}{C}\right)^n.$$

Therefore $\chi_n \to 0$ if $4\varepsilon < C$, which proves the Proposition. ∎

Remark 2.27 This Proposition can be strengthened somewhat. Namely, for any harmonic $\mathcal{E}nd_{\mathcal{B}}$-valued $(1,0)$-form θ_0 in a small neighborhood of 0 the series (2.8) converges to a form θ. As follows from [V-bun], the connection $\nabla + \theta$ is NHYM provided the following holds.

* All the forms $\theta_p \wedge \theta_q \in \mathcal{A}^{2,0}(M, \mathcal{E}nd\,\mathcal{B})$ lie in $\mathcal{A}^{2,0}_{\text{ex}}(M, \mathcal{E}nd\,\mathcal{B})$.

This condition is also knows as vanishing of all of the so-called Massey products $[\theta_0 \wedge \ldots \wedge \theta_0]$.

3 Autodual and NHYM connections in the hyperkähler case.

We now turn to the study of NHYM connections on hyperkähler manifolds. First we recall the definitions and some general facts.

Definition 3.1 ([C]) A **hyperkähler manifold** is a Riemannian manifold M equipped with two integrable almost complex structures I, J which are parallel with respect to the Levi-Civita connection and satisfy

$$I \circ J = -J \circ I.$$

Let M be a hyperkähler manifold. The operators I, J define an action of the quaternion algebra $\mathbb{H}$ on the tangent bundle TM. This action is also parallel. Every imaginary quaternion $a \in \mathbb{H}$ satisfying $a^2 = -1$ defines an almost complex structure on M. This almost complex structure is parallel, hence integrable and Kähler.

Definition 3.2 A complex structure on M corresponding to an imaginary quaternion $a \in \mathbb{H}$ with $a^2 = -1$ is said to be **induced by** a.

For every such $a \in \mathbb{H}$ we will denote by ω_a the Kähler form in the complex structure induced by a. We will always assume fixed a preferred complex Kähler structure I on M.

Recall that every hyperkähler manifold is equipped with a canonical holomorphic symplectic 2-form Ω. If $J, K \in \mathbb{H}$ satisfy $J^2 = -1$, $IJ = K$ then this form equals

$$\Omega = \omega_J + \sqrt{-1}\omega_K.$$

The group $U(\mathbb{H})$ of all unitary quaternions is isomorphic to $SU(2)$. Thus every hyperkähler manifold comes equipped with an action of $SU(2)$ on its tangent bundle, and, *a posteriori*, with an action of its Lie algebra $\mathfrak{su}(2)$. Extend these actions to the bundles $\Lambda^\bullet$ of differential forms and let $\Lambda^\bullet_{\text{inv}} \subset \Lambda^\bullet$ be the subbundle of $SU(2)$-invariant forms. The $SU(2)$-action does not commute with the de Rham differential. However, it does commute with the Laplacian (see [V-so5]). Therefore it preserves the subspace of harmonic forms. Identifying harmonic forms with coholomogy classes, we get an action of $SU(2)$ on the cohomology spaces $H^\bullet(M)$.

Let $\Lambda : \Lambda^{\bullet+2} \to \Lambda^\bullet$ be the Hodge operator associated to the Kähler metric on M.

Definition 3.3 A differential form θ on M is called **primitive** if $\Lambda\theta = 0$.

Lemma 3.4 *1. All $SU(2)$-invariant forms are primitive. All $SU(2)$-invariant 2-forms are of Hodge type $(1,1)$ for every one of the induced complex structures on M. Vice versa, if a form is of type $(1,1)$ for all the induced complex structures, it is $SU(2)$-invariant.*

2. The same statements hold for de Rham cohomology classes instead of forms.

Proof. See [V-bun], Lemma 2.1 ∎

Remark 3.5 The converse is true for $\dim_{\mathbb{C}} M = 2$, but in higher dimensiona there are primitive forms that are not $SU(2)$-invariant.

Consider a complex bundle $\mathcal{B}$ on M and let $\nabla : \mathcal{B} \to \mathcal{A}^1(\mathcal{B})$ be a connection on $\mathcal{B}$.

Definition 3.6 The connection ∇ is called **autodual** if its curvature $R \in \mathcal{A}^2(M, \mathfrak{End}\,\mathcal{B})$ is $SU(2)$-inavariant.

Remark 3.7 The terminology comes from the 4-dimensional topology: autodual connections on hyperkähler surfaces are anti-selfdual in the usual topological sense.

We will call an autodual connection ∇ $(0,1)$**-stable** if its $(1,0)$-part defines a stable holomorphic structure on the bundle $\mathcal{B}$. Denote by $\mathcal{M}^s_{\text{inv}}$ the set of equivalence classes of $(0,1)$-stable autodual connections on the bundle $\mathcal{B}$.

Remark 3.8 Like in Remark 2.4, this $(0,1)$-stability condition may be too restrictive.

Let ∇ be an autodual connection on $\mathcal{B}$. By Lemma 3.4 for every $J \in \mathbb{C}P^1$ the $(0,1)$-component of the connection ∇ with respect to the complex structure induced by J defines a holomorphic structure on $\mathcal{B}$. We will call it **the holomorphic structure induced by** J.

Proposition 3.9 *Let M be a hyperkähler manifold and let $\mathcal{B}$ be a complex bundle on M. Every autodual connection ∇ on $\mathcal{B}$ is NHYM.*

Proof. Immediately follows from Lemma 3.4. ∎

Therefore there exists a natural embedding $\mathcal{M}^s_{\text{inv}} \hookrightarrow \mathcal{M}^s$ from $\mathcal{M}^s_{\text{inv}}$ to the moduli space $\mathcal{M}^s$ of NHYM connections on $\mathcal{B}$. In the rest of this section we give a partial description of the image of this embedding.

We will use the following.

Theorem 3.10 *Assume that the first two Chern classes*

$$c_1(\mathcal{B}), c_2(\mathcal{B}) \in H^\bullet(M)$$

of the bundle $\mathcal{B}$ are $SU(2)$-invariant. Then every Hermitian Yang-Mills connection ∇ on $\mathcal{B}$ is autodual.

Proof. See [V-bun]. ∎

Therefore, if the Chern classes $c_1(\mathcal{B}), c_2(\mathcal{B})$ are $SU(2)$-invariant, then the closed subset $\mathcal{M}^s_u \in \mathcal{M}^s$ of Hermitian Yang-Mills connections lies in $\mathcal{M}^s_{\text{inv}}$.

Theorem 3.11 *Let M be a hyperkähler manifold and let $\mathcal{B}$ be a complex vector bundle on M such that the Chern classes $c_1(\mathcal{B}), c_2(\mathcal{B})$ are $SU(2)$-invariant. Then subset $\mathcal{M}^s_{\text{inv}} \subset \mathcal{M}^s$ of autodual connections contains an open neighborhood of the subset $\mathcal{M}^s_u \subset \mathcal{M}^s$ of connections admitting a compatible Hermitian metric.*

Proof. Let ∇ be an Hermitian Yang-Mills connection. It is autodual by Theorem 3.10, and it is enough to show that every $\nabla_1 \in \mathcal{F}_\nabla$ sufficiently close to ∇ is also autodual.

Let $\theta = \nabla_1 - \nabla$ and let $\theta_n, n \geq 0$ be as in (2.8). By Propostion 2.26 we can assume that

$$\theta = \sum_k \theta_k.$$

It is enough to prove that $\overline{D}\theta$ is $SU(2)$-invariant. We will prove that $\overline{D}\theta_k$ is $SU(2)$-invariant for all $k \geq 0$. To do this, we use results of [V-bun].

Consider the operator

$$L^{\Omega} : \mathcal{A}^{\bullet,\bullet}(\mathfrak{End}\,\mathcal{B}) \to \mathcal{A}^{\bullet+2,\bullet}(\mathfrak{End}\,\mathcal{B})$$

given by multiplication by the canonical holomorphic 2-form Ω on M. By [V-so5] the operator

$$[L_{\Omega}, \Lambda] : \mathcal{A}^{\bullet,\bullet}(\mathfrak{End}\,\mathcal{B}) \to \mathcal{A}^{\bullet+1,\bullet-1}(\mathfrak{End}\,\mathcal{B})$$

coincides with the action of a nilpotent element in the Lie algebra $\mathfrak{su}(2)$. Therefore it is a derivation with respect to the algebra structure on the complex $\mathcal{A}^{\bullet,\bullet}(\mathfrak{End}\,\mathcal{B})$. Moreover, a $(1,1)$-form α is $SU(2)$-invariant if and only if $[L_{\Omega}, \Lambda]\alpha = 0$.

Let $\partial^J : \mathcal{A}^{\bullet,\bullet}(\mathfrak{End}\,\mathcal{B}) \to \mathcal{A}^{\bullet+1,\bullet}(\mathfrak{End}\,\mathcal{B})$ be the commutator

$$\partial^J = \left[\overline{D}, [L_{\Omega}, \Lambda]\right].$$

This map again is a derivation with respect to the algebra structure on $\mathcal{A}^{\bullet,\bullet}(\mathfrak{End}\,\mathcal{B})$, and it is enough to prove that $\partial^J \theta_n = 0$ for all $n \geq 0$. Moreover,

$$\partial^J = \left[\overline{D}, [L_{\Omega}, \Lambda]\right] = \left[L_{\Omega}, \left[\overline{D}, \Lambda\right]\right] + \left[\Lambda, \left[L_{\Omega}, \overline{D}\right]\right],$$

The second term is zero since Ω is holomorphic, the first term is $[L_{\Omega}, \sqrt{-1}D^*]$ by Kodaira identity. Hence D^* and ∂^J anticommute. Finally, the Laplacian $\Delta_J = \partial_J\partial_J^* + \partial_J^*\partial_J$ is proportional by [V-bun] to the Laplacian $\Delta = DD^* + D^*D$. Therefore the Laplacian Δ and the Green operator G also commute with ∂^J.

Now, the form θ_0 is by definition Δ-harmonic. Therefore it is also Δ_J-harmonic, and $\partial^J\theta_0 = 0$. To prove that $\partial^J\theta_n = 0$, use induction on n. By definition

$$\partial^J\theta_n = \partial^J D^* G \left(\sum_{0 \leq k < n} \theta_k \wedge \theta_{n-1-k} \right) =$$

$$= -D^* G \left(\sum_{0 \leq k < n} \partial^J\theta_k \wedge \theta_{n-1-k} + \theta_k \wedge \partial^J\theta_{n-1-k} \right).$$

The right hand side is zero by the inductive assumption.

Thus all $\partial^J\theta_k$ are zero, and all the $\overline{D}\theta_k$ are $SU(2)$-invariant, which proves the Theorem. ∎

4 Stable bundles over twistor spaces.

4.1 Introduction

To further study autodual connections on a bundle $\mathcal{B}$ over a hyperkähler manifold M, we need to introduce the so-called "twistor space" X for M. This is a certain non-Kähler complex manifold associated to M. Autodual connections give rise to holomorphic bundles on X by means of a construction known as "twistor transform". This construction turns out to be essentially invertible, thus providing additional information on the moduli space $\mathcal{M}^s_{\operatorname{inv}}$.

We develop the twistor transform machinery in the next section. In this section we give the necessary preliminaries: the definition and some properties of the twistor space X, and a discussion of the notion of stability for holomorphic bundles over X.

4.2 Twistor spaces

Let M be a hyperkähler manifold. Consider the product manifold $X = M \times S^2$. Embed the sphere $S^2 \subset \mathbb{H}$ into the quaternion algebra $\mathbb{H}$ as the subset of all quaternions J with $J^2 = -1$. For every point $x = m \times J \in X = M \times S^2$ the tangent space T_xX is canonically decomposed $T_xX = T_mM \oplus T_JS^2$. Identify $S^2 = \mathbb{C}P^1$ and let $I_J : T_JS^2 \to T_JS^2$ be the complex structure operator. Let $I_m : T_mM \to T_mM$ be the complex structure on M induced by $J \in S^2 \subset \mathbb{H}$.

The operator $I_x = I_m \oplus I_J : T_xX \to T_xX$ satisfies $I_x \circ I_x = -1$. It depends smoothly on the point x, hence defines an almost complex structure on X. This almost complex structure is known to be integrable (see [Sal]).

Definition 4.1 The complex manifold $\langle X, I_x\rangle$ is called **the twistor space** for the hyperkähler manifold M.

By definition the twistor space comes equipped with projections $\sigma : X \to M$, $\pi : X \to \mathbb{C}P^1$. The second projection is holomorphic. For any point $m \in M$ the section $\widetilde{m} : \mathbb{C}P^1 \to X$ with image $m \times \mathbb{C}P^1 \subset X$ is also holomorphic. We will call this section $\widetilde{m}$ **the horizontal twistor line** corresponding to $m \in \mathcal{M}^s$.

Let $\iota : \mathbb{C}P^1 \to \mathbb{C}P^1$ be the real structure on $\mathbb{C}P^1$given by the antipodal involution. Then the product map

$$\iota = \operatorname{id} \times \iota : X \to X$$

defines a real structure on the complex manfiold X. The following fundamental property of twistor spaces is proved, e.g., in [HKLR].

Theorem 4.2 *Let M be a hyperkähler manifold and let X be its twistor space. Then a holomorphic section $\mathbb{C}P^1 \to X$ of the natural projection $\pi : X \to \mathbb{C}P^1$ is a horizontal twistor line if and only if it commutes with natural real structure $\iota : X \to X$.*

Let Sec be the Douady moduli space of holomorphic sections $\mathbb{C}P^1 \to X$ of the projection $\pi : X \to M$. Then conjugation by ι defines a real structure on the complex-analytic space Sec. Theorem 4.2 identifies the susbet of real points of Sec with the hyperkähler manifold M. We will call arbitrary holomorphic sections $\mathbb{C}P^1 \to X$ **twistor lines** in X.

4.3 Li–Yau theorem

The twistor space, in general, does not admit a Kähler metric. In order to obtain a good moduli space for holomorphic bundles on X we use a genralization of the notion of stability introduced by Li and Yau in [LY]. We reproduce here some of their results for the convenience of the reader.

Let X be an n-dimensional Riemannian complex manifold and let $\sqrt{-1}\omega$ be the imaginary part of the metric on X. Thus ω is a real $(1,1)$-form.

Assume that the form ω satisfies the following condition.

$$\omega^{n-2} \wedge d\omega = 0. \tag{4.9}$$

For a closed real 2-form η let

$$\deg \eta = \int_X \omega^{n-1} \wedge \eta.$$

The condition (4.9) ensures that $\deg \eta$ depends only on the cohomology class of η. Thus it defines a degree functional $\deg : H^2(X, \mathbb{R}) \to \mathbb{R}$. This functional allows one to repeat verbatim the Mumford-Takemoto definitions of stable and semistable bundles in this more general situation. Moreover, the Hermitian Yang-Mills equations also carry over word-by-word.

Yau and Li proved the following.

Theorem 4.3 ([LY]) *Let X be a complex Riemannian manifold satisfying (4.9). Then every stable holomorpic bundle $\mathcal{B}$ on X admits a unique Hermitian Yang-Mills connection ∇. Vice versa, every bundle $\mathcal{B}$ admitting an Hermitian Yang-Mills connection is polystable.*

Just like in the Kähler case, this Theorem allows one to construct a good moduli space for holomorphic bundles on X. (See [Kob].)

4.4 Li–Yau condition for twistor space

The twistor space $X = M \times \mathbb{C}P^1$ is equipped with a natural Riemannian metric, namely, the product of the metrics on M and on $\mathbb{C}P^1$. To apply the Li-Yau theory to X, we need to check the condition (4.9). First, we identify the form ω.

Let $\omega = \omega_M + \omega_{\mathbb{C}P^1}$ be the decomposition associated with the product decomposition $X = M \times \mathbb{C}P^1$. By the definition of the complex structure on X, the form $\omega_{\mathbb{C}P^1}$ is the pullback $\pi^*\omega$ of the usual Kähler form on $\mathbb{C}P^1$, while ω_M is a certain linear combination of pullbacks of Kähler forms on M associated to different induced complex structures.

Let $W \in \mathbb{H}$ be the 3-dimensional subspace of imaginary quaternions. For every $a \in W$ the metric $\langle \cdot, \cdot \rangle$ on the hyperkähler manifold M defines a real closed 2-form ω_a on M by the rule

$$\omega_a(\cdot, \cdot) = \langle \cdot, a\cdot \rangle.$$

This construction is linear in a, hence defines an embedding $W \hookrightarrow \mathcal{A}^2(M, \mathbb{R})$. Let $\mathcal{W}$ be the trivial bundle on $\mathbb{C}P^1$ with the fiber W. The embedding $W \hookrightarrow \mathcal{A}^2(M, \mathbb{R})$ extends to an embedding

$$\mathcal{W} \hookrightarrow \pi_*\sigma^*\mathcal{A}^2(M, \mathbb{R}) \subset \pi_*\mathcal{A}^2(X).$$

Since $X = M \times \mathbb{C}P^1$, the de Rham differential $d = d_X : \mathcal{A}^\bullet(X) \to \mathcal{A}^{\bullet+1}(X)$ decomposes into the sum $d = d_M + d_{\mathbb{C}P^1}$. The differential $d_{\mathbb{C}P^1}$ defines a flat connection on the bundle $\pi_*\sigma^*\mathcal{A}^2(M, \mathbb{R})$. The subbundle $\mathcal{W} \subset \pi_*\sigma^*\mathcal{A}^2(M, \mathbb{R})$ is flat with respect to $d_{\mathbb{C}P^1}$.

The space W is equipped with an euclidian metric, thus $W = W^*$. Since we have an embedding $\mathbb{C}P^1 = S^2 \hookrightarrow W$, the bundle $\mathcal{W} = TW|_{\mathbb{C}P^1} = T^*W|_{\mathbb{C}P^1}$ decomposes orthogonally

$$\mathcal{W} = \mathbb{R} \oplus \mathcal{O}(-2)$$

into the sum of the conormal and the cotangent bundles to $\mathbb{C}P^1 \subset W$. The conormal bundle is the trivial 1-dimesional real bundle $\mathbb{R}$, and the cotangent bundle is isomorphic to the complex vector bundle $\mathcal{O}(-2)$ on $\mathbb{C}P^1$. The connection $d_x|_{\mathcal{W}}$ induces the trivial connection on $\mathbb{R}$ and the usual metric connection on $\mathcal{O}(2)$. The embedding $\mathcal{W} \to \pi_*\sigma\mathcal{A}^2(M, \mathbb{R})$ decomposes then into a real 2-form

$$\omega \in \sigma^*\mathcal{A}^2(M, \mathbb{R})$$

and a complex $\mathcal{O}(2)$-valued 2-form

$$\Omega \in \sigma^*\mathcal{A}^2(M, \mathbb{R}) \otimes \pi^*\mathcal{O}(2).$$

The form Ω is holomorphic, while the form $\sqrt{-1}\omega_M$ is precisely the imaginary part of the Hermitian metric on X.

Let now $\upsilon \in \mathcal{A}^{2,1}(X)$ be the $(2,1)$-form corresponding to the holomorphic form

$$\Omega \in \mathcal{A}^{2,0}(X, \pi^*\mathcal{O}(2))$$

under the identification $\mathcal{O}(2) \cong \mathcal{A}^{0,1}(\mathbb{C}P^1)$ provided by the metric on $\mathbb{C}P^1$.

Lemma 4.4

$$d\omega = \upsilon + \overline{\upsilon}.$$

Proof. Indeed, $d\omega_{\mathbb{C}P^1} = 0$ and $d_M\omega_M = 0$, therefore it is enough to compute $d_{\mathbb{C}P^1}\omega_M$. The bundle $\mathcal{W} \in \mathcal{A}^2(X)$ is invariant under $d_{\mathbb{C}P^1}$, and d induces the trivial connection ∇ on $\mathcal{W}$. Let

$$d = \begin{bmatrix} \nabla_{\mathbb{R}} & \theta_{10} \\ \theta_{01} & \nabla_{\mathcal{O}(2)} \end{bmatrix}$$

be the decomposition of ∇ with respect to $\mathcal{W} = \mathbb{R} \oplus \mathcal{O}(2)$. The connection $\nabla_{\mathbb{R}}$ is trivial, therefore $\nabla_{\mathbb{R}}\omega = 0$. An easy computation shows that

$$\theta_{10} \in \mathcal{A}^1(\mathbb{C}P^1, \mathcal{O}(2))$$

induces the isomorphism $\mathcal{A}^1(\mathbb{C}P^1) \cong \mathcal{O}(2)$. Thus

$$d\omega_M = d_{\mathbb{C}P^1}\omega_M = \theta_{10} \wedge \Omega = \upsilon + \overline{\upsilon}.$$

∎

Now we can prove that the twistor space X satisfies the condition (4.9).

Proposition 4.5 *The canonical $(1,1)$-form ω on the twistor space X satisfies*

$$\omega^{n-1} \wedge d\omega = 0$$

for $n = \dim M = \dim X - 1$.

Proof. Let $x = m \times J \in X$ be a point in X. Choose a local coordinate z on $\mathbb{C}P^1$ near the point $J \in \mathbb{C}P^1$. In a neighborhood of $x \in X$ we have

$$\upsilon = f(z)\Omega \wedge d\bar{z}$$

for some holomorphic function $f(z)$. Therefore

$$\omega^{n-1} \wedge d\omega = f(z)\omega^{n-1} \wedge \Omega \wedge d\bar{z} + \overline{f(z)}\omega^{n-1} \wedge \overline{\Omega} \wedge dz.$$

But $\omega^{n-1} \wedge \Omega$ and $\omega \wedge \overline{\Omega}$ are both forms on the n-dimensional manifold M, of Hodge types $(n+1, n-1)$ and $(n-1, n+1)$. Hence both are zero. To prove the Proposition, it remains to see that $\omega^{n-1} - \omega_M^{n-1}$ is divisible by $\omega_{\mathbb{C}P^1}$, and $\omega_{\mathbb{C}P^1} \wedge dz = \omega_{\mathbb{C}P^1} \wedge d\bar{z} = 0$. ∎

5 Twistor transform.

5.1 Twistor transform

We now introduce the direct and inverse twistor transforms which relate autodual bundles on the hyperkähler manifold M and holomorphic bundles on its twistor space X.

Let $\mathcal{B}$ be a complex vector bundle on M equipped with a connection ∇. The pullback $\sigma^*\mathcal{B}$ of $\mathcal{B}$ to X is then equipped with a connection $\sigma^*\nabla$.

Lemma 5.1 *The connection ∇ is autodual if and only if the connection $\sigma^*\nabla$ has curvature of Hodge type $(1,1)$.*

Proof. Indeed, the curvature R_X of $\sigma^*\nabla$ is equal to the pullback σ^*R_M of the curvature R_M of ∇. Therefore it is of Hodge type $(1,1)$ on X if and only if for every $I \in \mathbb{C}P^1$ the form R_M is of type $(1,1)$ in the induced complex structure I. By Lemma 3.4 this happens if and only if R_M is $SU(2)$-invariant. ∎

In particular, for every autodual bundle $\langle \mathcal{B}, \nabla \rangle$ the $(0,1)$-component $\sigma^*\nabla^{0,1}$ of the connection $\sigma^*\nabla$ satisfies $\sigma^*\nabla^{0,1} \circ \sigma^*\nabla^{0,1} = 0$ and defines a holomorphic structure on the bundle $\sigma^*\mathcal{B}$.

Definition 5.2 The holomorphic bundle $\langle \sigma^*\mathcal{B}, \sigma^*\nabla^{0,1} \rangle$ is called **the twistor transform** of the autodual bundle $\langle \mathcal{B}, \nabla \rangle$.

5.2 $\mathbb{C}P^1$-holomorphic bundles over twistor spaces

The twistor transform is in fact invertible. To construct an inverse transform, we begin with some results on differential forms on the twistor space X.

The product decomposition $X = M \times \mathbb{C}P^1$ induces the decomposition $\mathcal{A}^1(X) = \sigma^*\mathcal{A}^1(M) \oplus \pi^*\mathcal{A}^1(\mathbb{C}P^1)$ of the bundle $\mathcal{A}^1(M)$ of 1-forms. By the definition of the complex structure on X, the projection onto the subbundle of $(0,1)$-forms commutes with the projection onto the bundle $\pi^*\mathcal{A}^1(\mathbb{C}P^1)$. Therefore a Dolbeault differential

$$\bar\partial_{\mathbb{C}P^1} : \mathcal{A}^0(X) \to \pi^*\mathcal{A}^{0,1}(\mathbb{C}P^1)$$

is well-defined.

Definition 5.3 A $\mathbb{C}P^1$**-holomorphic bundle** on X is a complex vector bundle $\mathcal{B}$ on X equipped with an operator $\bar\partial_{\mathbb{C}P^1} : \mathcal{B} \to \mathcal{B} \otimes \pi^*\mathcal{A}^{0,1}(\mathbb{C}P^1)$ satisfying

$$\bar\partial_{\mathbb{C}P^1}(fa) = \bar\partial_{\mathbb{C}P^1}(f)a + f\bar\partial_{\mathbb{C}P^1}(a)$$

for a function f and a local section a of $\mathcal{B}$.

Remark 5.4 For any point $m \in M$ the restriction $\widetilde{m}^*\mathcal{B}$ of a $\mathbb{C}P^1$-holomrphic bundle $\mathcal{B}$ to the horizontal twistor line $\widetilde{m} : \mathbb{C}P^1 \to X$ is holomorphic in the usual sense.

A $\bar\partial_{\mathbb{C}P^1}$-closed smooth section $a \in \Gamma(X, \mathcal{B})$ of a $\mathbb{C}P^1$-holomorphic bundle $\mathcal{B}$ will be called $\mathbb{C}P^1$**-holomorphic**. Tensor products and $\mathcal{H}om$-bundles of $\mathbb{C}P^1$-holomorphic bundles are again $\mathbb{C}P^1$-holomorphic. A differential operator $f : \mathcal{B}_0 \to \mathcal{B}_1$ will be called $\mathbb{C}P^1$**-holomorphic** if $\bar\partial f(a) = 0$ for every local $\mathbb{C}P^1$-holomorphic section a of the bundle $\mathcal{B}$.

For every complex vector bundle $\mathcal{B}$ on M the bundle $\sigma^*\mathcal{B}$ on X is canonically $\mathbb{C}P^1$-holomorphic. For a $\mathbb{C}P^1$-holomorphic bundle $\mathcal{B}$ let $\sigma_*\mathcal{B}$ be the sheaf on M of $\mathbb{C}P^1$-holomorphic sections of $\mathcal{B}$. The functors σ_* and σ^* are adjoint.

Definition 5.5 A $\mathbb{C}P^1$-holomorphic bundle $\mathcal{B}$ on X is called $\mathbb{C}P^1$**-constant** if it is isomorphic to $\sigma^*\mathcal{B}_1$ for a complex vector bundle $\mathcal{B}$ on M.

Since $\sigma_*\sigma^*\mathcal{B}_1 \cong \mathcal{B}_1$ for every complex bundle $\mathcal{B}_1$ on M , a bundle $\mathcal{B}$ on X is $\mathbb{C}P^1$-constant if and only if the canonical map $\mathcal{B} \to \sigma^*\sigma_*\mathcal{B}$ is an isomorphism. The functor σ^* is therefore an equivalence between the category of complex vector bundles on M and the category of $\mathbb{C}P^1$-constant $\mathbb{C}P^1$-holomorphic bundles on X.

Lemma 5.6 *A $\mathbb{C}P^1$-holomorphic bundle $\mathcal{B}$ on X is $\mathbb{C}P^1$-constant if and only if for every horizontal twistor line $\widetilde{m} : \mathbb{C}P^1 \to X$ the restriction $\widetilde{m}^*\mathcal{B}$ is trivial.*

Proof. Clear. ∎

Note that if $\dim \Gamma(\mathbb{C}P^1, \widetilde{m}^* \mathcal{B})$ is the same for every horizontal twistor line $\widetilde{m} : \mathbb{C}P^1 \to X$, then $\sigma_* \mathcal{B}$ is the sheaf of smooth sections of a complex vector bundle on M.

5.3 Differential forms and $\mathbb{C}P^1$-holomorphic bundles

Let $\mathcal{A}^\bullet_M(X) = \sigma^* \mathcal{A}^\bullet(M) \subset \mathcal{A}^\bullet(X)$ be the subcomplex of relative $\mathbb{C}$-valued forms on X over $\mathbb{C}P^1$ and let $\mathcal{A}^{0,\bullet}_M(X)$ be the quotient complex of forms of Hodge type $(0, \bullet)$. Denote by d_M and $\bar\partial_M$ the corresponding differentials and let $P : \mathcal{A}^\bullet_M \to \mathcal{A}^{0,\bullet}_M$ be the natural projection. By definition all the bundles $\mathcal{A}^\bullet_M$ and $\mathcal{A}^{0,\bullet}_M$ are $\mathbb{C}P^1$-holomorphic, and the differentials d_M and $\bar\partial_M$ are $\mathbb{C}P^1$-holomorphic.

Let $\mathcal{B}$ be a holomorphic bundle on X. Then the complex structure operator $\bar\partial : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}^{0,1}(X, \mathcal{B})$ can be decomposed

$$\bar\partial = \bar\partial_M + \bar\partial_{\mathbb{C}P^1}$$

into an operator $\partial_{\mathbb{C}P^1} : \mathcal{B} \to \otimes \mathcal{B} \otimes \pi^* \mathcal{A}^{0,1}(\mathbb{C}P^1)$ and an operator $\bar\partial_M : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}^{0,1}_M$. The operator $\bar\partial_{\mathbb{C}P^1}$ is a $\mathbb{C}P^1$-holomorphic structure on the bundle $\mathcal{B}$, and the operator $\bar\partial_M$ is $\mathbb{C}P^1$-holomorphic.

This constrution is in fact invertible. Namely, we have the following.

Lemma 5.7 *The correspondence $\mathcal{B} \mapsto \langle \mathcal{B}, \bar\partial_M \rangle$ is an equivalence between the category of holomorphic bundles on X and the category of $\mathbb{C}P^1$-holomorphic bundles $\mathcal{B}$ on X equipped with a $\mathbb{C}P^1$-holomorphic operator $\bar\partial_M$: $\mathcal{B} \to \mathcal{B} \otimes \mathcal{A}^{0,1}_M$ satisfying $0 = \bar\partial_M^2 : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}^{0,2}_M$ and*

$$\bar\partial_M(fa) = \bar\partial(f)a + f\bar\partial_M(a)$$

for a function f and a local section a of $\mathcal{B}$.

Proof. Clear. ∎

In particlular, every holomorphic bundle $\mathcal{B}$ on X is canonically $\mathbb{C}P^1$-holomorphic. We will call a holomorphic bundle $\mathcal{B}$ on X **$\mathbb{C}P^1$-constant** if the corresponding $\mathbb{C}P^1$-holomorphic bundle is $\mathbb{C}P^1$-constant in the sense of Definition 5.5.

5.4 The complex $\mathcal{A}^\bullet_{top}(M)$: definition.

For any horizontal twistor line $\widetilde{m} : \mathbb{C}P^1 \to X$ the restriction $\widetilde{m}^*\mathcal{A}^i_M$ is trivial, while $\widetilde{m}^*\mathcal{A}^{0,i}_M$ is a sum of several copies of the bundle $\mathcal{O}(i)$ on $\mathbb{C}P^1$ (see, e.g., [HKLR]). Therefore $\sigma_*\mathcal{A}^*{}^\bullet{}_M \cong \mathcal{A}^\bullet(M,\mathbb{C})$, and the map P induces a projection

$$P : \mathcal{A}^\bullet(M,\mathbb{C}) \cong \sigma_*\mathcal{A}^\bullet_M(X) \to \sigma_*\mathcal{A}^{0,1}_M(X).$$

Let $\mathcal{A}^{1,0}_M(X)$ be the sheaf of relative forms of type $(1,0)$. By definition we have an exact sequence

$$0 \to \mathcal{A}^{1,0}_M(X) \to \mathcal{A}^1_M(X) \to \mathcal{A}^{0,1}_M(X) \to 0$$

of $\mathbb{C}P^1$-holomorphic bundles on X. Consider the associated long exact sequence for σ_*. The restriction of $\mathcal{A}^{1,0}_M$ to any horizontal twistor line $\widetilde{m} : \mathbb{C}P^1 \to X$ is a sum of several copies of $\mathcal{O}(-1)$. Therefore this long sequence reduces to the map

$$\sigma_*\mathcal{A}^1_M(X) \xrightarrow{P} \sigma_*\mathcal{A}^{0,1}_M(X),$$

which is therefore an isomorphism.

Recall that the bundle $\mathcal{A}^i(M)$ carries a representation of the group $SU(2)$ for every $i \geq 0$. This representation is completely reducible [8], and contains isotypical components of highest weights $\leq i$. Let $\mathcal{A}^i_{\text{top}} \subset \mathcal{A}^i$ be the component of highest weight exactly i.

Lemma 5.8 *The map $P : \mathcal{A}^\bullet(M,\mathbb{C}) \to \sigma_*\mathcal{A}^{0,\bullet}_M(X)$ is compatible with the $SU(2)$-action on $\mathcal{A}^\bullet(M,\mathbb{C})$. The restriction $P\mathcal{A}^\bullet_{\text{top}}(M,\mathbb{C}) \to \sigma_*\mathcal{A}^{0,\bullet}_M$ is an isomorphism.*

Proof. We already know the claim for $\sigma_*\mathcal{A}^{0,1}_M$. Let $i > 1$. Note that $\sigma_*\mathcal{A}^{0,i}_M$ is equipped with an $SU(2)$-action by the Borel-Weyl theory. The corresponding representation is of highest weight i. The map P is obviously compatible with this action, which proves the first statement. To prove the second, it is enough to prove that P is invertible on the subbundles of highest vectors. But both these subsundles equal $\mathcal{A}^{0,i}(M)$. $\blacksquare$

[8] $SU(2)$ acts along the fibers, which are finite-dimensional

5.5 The complex $\mathcal{A}^{\bullet}_{top}(M)$ and autodual bundles.

The complex $\mathcal{A}^{\bullet}_{\mathrm{top}}(M) \cong \sigma_*\mathcal{A}^{0,1}_M$ plays the same role for autodual bundles as the Dolbeault resp. de Rham complexes play for holomorphic resp. flat ones. Precisely, let $\mathcal{B}$ be a complex vector bundle on M equipped with a connection $\nabla : \mathcal{B} \to \mathcal{A}^1(\mathcal{B})$. Extend the operator ∇ to a differential operator $D : \mathcal{A}^{\bullet}_{\mathrm{top}}(\mathcal{B}) \to \mathcal{A}^{\bullet+1}_{\mathrm{top}}(\mathcal{B})$ by means of the embedding $\mathcal{A}^{\bullet}(\mathcal{B}) \hookrightarrow \mathcal{A}^{\bullet}(\mathcal{B})$ and the natural $SU(2)$-invariant projection $\mathcal{A}^{\bullet+1}(\mathcal{B}) \to \mathcal{A}^{\bullet+1}_{\mathrm{top}}(\mathcal{B})$.

Lemma 5.9 *The connection ∇ is autodual if and only if its extension D satisfies $D^2 = 0$.*

Proof. The operator D^2 is the multiplication by the $\mathcal{A}^2_{\mathrm{top}}$-part of the curvature R of the bundle $\mathcal{B}$ with respect to the decomposition

$$\mathcal{A}^2(\mathcal{E}\mathfrak{nd}\,\mathcal{B}) = \mathcal{A}^2_{\mathrm{top}}(\mathcal{E}\mathfrak{nd}\,\mathcal{B}) \oplus \mathcal{A}^2_{\mathrm{inv}}(\mathcal{E}\mathfrak{nd}\,\mathcal{B}).$$

Thus it vanishes if and only if R is $SU(2)$-invariant, which by definition means that ∇ is autodual. ∎

Let $\mathcal{B}$ be a complex vector bundle equipped with an autodual connection ∇. Since $\mathcal{A}^1(\mathcal{B}) \cong \sigma_*\mathcal{A}^{0,1}_M(X) \times \mathcal{B}$, the map $\nabla : \mathcal{B} \to \mathcal{B} \otimes \sigma_*\mathcal{A}^{0,1}_M(X)$ defines a $\mathbb{C}P^1$-holomorphic map

$$\bar\partial_M : \sigma^*\mathcal{B} \to \sigma^*\mathcal{B} \otimes \mathcal{A}^{0,1}_M(X)$$

of $\mathbb{C}P^1$-holomorphic bundles on X. By Lemmas 5.8 and 5.9 the map $\bar\partial_M$ extends to a map $\mathcal{A}^{0,\bullet}_M(X) \otimes \sigma^*\mathcal{B} \to \mathcal{A}^{0,\bullet+1}_M(X) \otimes \mathcal{B}$ satisfying $\bar\partial_M^2 = 0$. By Lemma 5.7 this map defines a holomorphic structure on the bundle $\sigma^*\mathcal{B}$.

Lemma 5.10 *The holomorphic bundle $\langle \sigma^*\mathcal{B}, \bar\partial_M \rangle$ on X is isomorphic to the twistor transform of the autodual bundle $\mathcal{B}$.*

Proof. Clear. ∎

Let now $\mathcal{B}$ be an arbitrary $\mathbb{C}P^1$-constant holomorphic bundle on X. Then the sheaf $\sigma_*\mathcal{B}$ is the sheaf of sections of a vector bundle. Since $\mathcal{B}$ is $\mathbb{C}P^1$-constant, $\mathcal{B} \cong \sigma^*\sigma_*\mathcal{B}$. The operator

$$\partial_M : \mathcal{B} \cong \sigma^*\sigma_*\mathcal{B} \to \mathcal{B} \otimes \mathcal{A}^{0,1}_M(X)$$

gives by adjuction an operator

$$\nabla : \sigma_*\mathcal{B} \to \sigma_*\left(\mathcal{B} \otimes \mathcal{A}^{0,1}_M(X)\right) \cong \sigma_*\mathcal{B} \otimes \mathcal{A}^1_{\mathrm{top}}(M).$$

By Lemmas 5.8 and 5.9 the operator ∇ is an autodual connection on $\sigma_*\mathcal{B}$.

Definition 5.11 The autodual bundle $\langle \sigma_*, \nabla \rangle$ on M is called **the inverse twistor transform** of the $\mathbb{C}P^1$-constant holomorphic bundle $\mathcal{B}$ on X.

Theorem 5.12 *The direct and inverse twistor transforms are mutually inverse equivalences between the category of autodual bundles on M and the category of $\mathbb{C}P^1$-constant holomorphic bundles on X.*

Proof. Clear. ∎

6 Stability of the twistor transform.

6.1 Introduction

Let M be a hyperkähler manifold and let X be its twistor space. Consider a semistable autodual bundle $\langle \mathcal{B}, \nabla \rangle$ on M and let $\sigma^*\mathcal{B}$ be its twistor transform. The bundle $\sigma^*\mathcal{B}$ is a holomorphic bundle on X. In this section we prove under certain conditions that $\sigma^*\mathcal{B}$ is semistable in the sense of 4.3. More precisely, we have the following.

Proposition 6.1 *Let M be a hyperkähler manifold. Denote its twistor space by X. Let $\langle \mathcal{B}, \nabla \rangle$ be a semistable autodual bundle on M and let $\sigma^*\mathcal{B}$ be its twistor transform. Then for every coherent subsheaf $\mathcal{F} \subset \sigma^*\mathcal{B}$ we have*

$$\frac{\deg c_1(\mathcal{F})}{\operatorname{rank} \mathcal{F}} \leq \frac{\deg c_1(\sigma^*\mathcal{B})}{\operatorname{rank} \sigma^*\mathcal{B}},$$

where $\operatorname{rank} \mathcal{F}$ *is the rank of the generic fiber of* $\mathcal{F}$ *and* $\deg$ *is understood in the sense of subsection 4.3.*

6.2 Semistability for $\mathbb{C}P^1$-constant bundles

Before we give a proof of this Proposition, we prove the following.

Lemma 6.2 *Let $\mathcal{B}$ be a holomorphic bundle on the twistor space X. Assume that $\mathcal{B}$ is $\mathbb{C}P^1$-constant (that is, for every horizontal twistor line $\widetilde{m} : \mathbb{C}P^1 \to X$ the restriction $\widetilde{m}^*\mathcal{B}$ is trivial). Then $\deg \mathcal{B} = 0$, and the bundle $\mathcal{B}$ is semistable.*

Proof. Since $H^1(\mathbb{C}P^1, \mathbb{R}) = 0$,

$$H^2(X, \mathbb{R}) = H^2(M, \mathbb{R}) \oplus H^2(\mathbb{C}P^1, \mathbb{R}). \tag{6.10}$$

Since $\mathcal{B}$ is $\mathbb{C}P^1$-constant, $c_1(\mathcal{B}) \in H^2(M,\mathbb{R})$. For every $I \in \mathbb{C}P^1$ let $X_I = \pi^{-1}(I) \subset X$ be the fiber over I. Since

$$c_1(\mathcal{B})_{X_I} = c_1(\mathcal{B}|_{X_I}) \in H^{1,1}_I(M)$$

is of Hodge type $(1,1)$ for every $I \in \mathbb{C}P^1$, the class $c_1(\mathcal{B}) \in H^2(M,\mathbb{R})$ is $SU(2)$-invariant by Lemma 3.4. Therefore $\deg c_1(\mathcal{B}) = \Lambda c_1(\mathcal{B}) = 0$.

To prove semistability, let $\mathcal{F} \subset \mathcal{B}$ be a coherent subsheaf. It is enough to prove that $\deg c_1(\mathcal{F}) \leq 0$. Let $c_1(\mathcal{F}) = c_M + c_{\mathbb{C}P^1}$ be the decomposition associated with (6.10). Again, by Lemma 3.4 c_M is $SU(2)$-invariant, and $\deg c_1(\mathcal{F}) = \deg c_{\mathbb{C}P^1}$. For a generic horizontal twistor line $\widetilde{m} : \mathbb{C}P^1 \to X$ we have $c_{\mathbb{C}P^1} = c_1(\widetilde{m}^*\mathcal{F}) \in H^2(\mathbb{C}P^1,\mathbb{R})$. Since $\widetilde{m}^*\mathcal{B}$ is trivial, it is semistable, and $\deg c_1(\widetilde{m}^*\mathcal{F}) \leq 0$. ∎

Let $\mathcal{M}^{ss}_X$ be the moduli space of semistable holomorphic bundles on X. Lemma 6.2 implies that the set $\mathcal{M}^s_{const}$ of equivalence classes of $\mathbb{C}P^1$-constant holomorphic bundles on X is a subset of $\mathcal{M}^{ss}_X$. The subset $\mathcal{M}^s_{const} \subset \mathcal{M}^{ss}_X$ is open.

6.3 Conclusion

The Propostion now follows directly from Lemma 6.2. Moreover, the twistor transform provides an isomorphism $\mathcal{M}^s_{\mathrm{inv}} \to \mathcal{M}^s_{const} \subset \mathcal{M}^{ss}_X$ from the moduli space $\mathcal{M}^s_{\mathrm{inv}}$ of autodual bundles on M to the open subset of $\mathbb{C}P^1$-constant bundles in the moduli space $\mathcal{M}^{ss}_X$

7 Stable bundles and projective lines in twistor spaces.

7.1 Hyperkähler structure on the Mukai dual space

Let M be a compact hyperkähler manifold and let $\mathcal{B}$ be a complex vector bundle on M with $SU(2)$-invariant Chern classes $c_1(\mathcal{B})$ and $c_2(\mathcal{B})$. Consider the moduli space $\mathcal{M}^s_0$ of stable holomorphic structures on $\mathcal{B}$ and let $\mathcal{M}^{reg}_0 \subset \mathcal{M}^s_0$ be the dense open subset of smooth points in $\mathcal{M}^s_0$. Recall that the subset $\mathcal{M}^{reg}_0$ is equipped with a natural Kähler metric, called *the Weil-Peterson metric.*

It was proved in [V-bun] that the Weil-Peterson metric on $\mathcal{M}^{reg}_0$ is actually hyperkähler. Moreover, the complex manifold $(\mathcal{M}^{reg}_0)_J$ with the complex structure induced by a quaternion $J \in \mathbb{C}P^1 \subset \mathbb{H}$ was naturally identified with the subset of smooth points in the moduli space of stable holomorphic structures on $\mathcal{B}$ with respect to the complex structure J on M.

Let $\mathcal{X}_{reg}$ be the twistor space of the hyperkähler manifold $\mathcal{M}_0^{reg}$. Consider the topological space $\mathcal{X} = \mathcal{M}_0^s \times \mathbb{C}P^1$. We have a natural embedding $\mathcal{X}_{reg} \subset X$. In [V-bun] the complex structure on $\mathcal{X}_{reg}$ and the real structure $\iota : \mathcal{X}_{reg} \to \mathcal{X}_{reg}$ were naturally extended to the whole of $\mathcal{X}$. The complex-analytic space $\mathcal{X}$ is in general singular. However, the fundamental Theorem 4.2 still holds for the natural projection $\pi : \mathcal{X} \to \mathbb{C}P^1$. We will call holomorphic sections $\mathbb{C}P^1 \to \mathcal{X}$ of the projection $\pi : \mathcal{X} \to \mathbb{C}P^1$ **twistor lines** in $\mathcal{X}$. The space $\mathcal{M}_0^s$ is then naturally isomorphic to the subset of real twistor lines in $\mathcal{X}$.

These data define **singular hyperkähler structure** on $\mathcal{M}_0^s$ (see [V-bun] for details). The space $\mathcal{M}_0^s$ with this hyperkähler structure is called **Mukai dual** to M (results of [V-bun] generalise Mukai's work about duality of K3 surfaces). We must caution the reader that this version of Mukai duality is not involutive, as the term "dual" might erroneously imply.

7.2 Fiberwise stable bundles

Let X be the twistor space of the hyperkähler manifold M. Let $\mathcal{M}^s_{const} \subset \mathcal{M}^{ss}_X$ be the open subset of $\mathbb{C}P^1$-constant holomorphic structures in the moduli space $\mathcal{M}^{ss}_X$ of semistable holomorphic structures on the bundle $\sigma^*\mathcal{B}$. In the last section we have identified $\mathcal{M}^s_{const}$ with the space $\mathcal{M}^s_{\mathrm{inv}}$ of $(0,1)$-stable autodual connections on the bundle $\mathcal{B}$.

In this section we will need still another notion of stability for holomorphic bundles over X.

Definition 7.1 Call a stable holomorphic structure $\bar\partial$ on $\sigma^*\mathcal{B}$ **fiberwise stable** if for any $L \in \mathbb{C}P^1$ the restriction of $\langle \sigma^*\mathcal{B}, \bar\partial\rangle$ to the fiber $X_L = \pi^{-1}(L) \subset X$ is stable.

Let $\mathcal{M}^s_{\mathrm{fib}} \subset \mathcal{M}^{ss}_X$ be the subset of fiberwise stable holomorphic structures. The intersection $(\mathcal{M}^s_{const} \cap \mathcal{M}^s_{\mathrm{fib}}) \subset \mathcal{M}^{ss}_X$ is, then, isomorphic to the moduli space of autodual connections on $\mathcal{B}$ inducing a stable holomorphic structure on $\mathcal{B}$ for every $I \in \mathbb{C}P^1$.

The goal of this section is to prove the following.

Theorem 7.2 *The space $\mathcal{M}^s_{\mathrm{fib}}$ is naturally isomorphic to the space* Sec *of twistor lines in the manifold $\mathcal{X}$.*

7.3 Stability of fiberwise-stable bundles

We begin by noting that one of the conditions in Definition 7.1 is in fact redundant.

Lemma 7.3 *Let $\mathcal{B}$ be a holomorphic bundle on the twistor space X. If the restriction $i^*\mathcal{B}$ is stable for a generic*[9] *point $I \in \mathbb{C}P^1$, then the bundle $\mathcal{B}$ is stable.*

Proof. Indeed, it was proved in [V-sym] that for a generic point $I \in \mathbb{C}P^1$ every rational $(1,1)$-cohomology class for the fiber X_I is of degree zero. Therefore a stable holomorphic bundle on X_I has no proper subsheaves. Hence for a proper subsheaf $\mathcal{F} \subset \mathcal{B}$ either $\mathcal{F}$ or $\mathcal{B}/\mathcal{F}$ is supported on non-generic fibers of $\pi : X \to \mathbb{C}P^1$. In particular, either $\mathcal{F}$ or $\mathcal{B}/\mathcal{F}$ is a torsion sheaf. This implies that the bundle $\mathcal{B}$ is stable. ∎

7.4 Modular interpretation of the Mukai dual twistor space

We now construct a map $\mathcal{M}^s_{\text{fib}} \to \operatorname{Sec}$. To do this, we give a modular interpretation of the space $\mathcal{X}$.

For any point $I \in \mathbb{C}P^1$ let $i : X_I \hookrightarrow X$ be the natural embedding of the fiber $X_I = \pi^{-1}(I) \subset X$. For a stable holomorphic bundle $\mathcal{B}$ on the fiber X_I call the coherent sheaf $i_*\mathcal{B}$ on X **a stable sheaf on X supported in I**, or simply a **fiber-supported stable sheaf**.

More generally, for a complex analytic space Z call a coherent sheaf $\mathcal{E}$ on $Z \times X$ **a family of fiber-supported stable sheaves on X** if there exists a holomorphic map $f_Z : Z \to \mathbb{C}P^1$ such that $\mathcal{B}$ and a holomorphic bundle $\mathcal{E}_0$ on the subspace $Z \times_{\mathbb{C}P^1} X \subset Z \times X$ such that

1. $\mathcal{E} \cong i_*\mathcal{E}_0$, where $i : Z \times_{\mathbb{C}P^1} X \to Z \times X$ is the natural embedding.
2. For every point $z \in Z$ the restriction of $\mathcal{E}$ to $z \times \pi^{-1}(f_Z(z)) \subset Z \times X$ is a stable holomorphic bundle.

The space $\mathcal{X}$ is obviously the moduli space for families of fiber-supported stable sheaves on X. The holomorphic map $f_{\mathcal{X}} : \mathcal{X} \to \mathbb{C}P^1$ is the natural projection.

Let now $\mathcal{B}$ be a fiberwise stable holomorphic bundle on X. For every $I \in \mathbb{C}P^1$ the coherent sheaf $i_*i^*\mathcal{B}$ on X is a stable sheaf supported in I. The correspondence $I \mapsto i_*i^*\mathcal{B}$ defines a holomorphic map $\mathbb{C}P^1 \to \mathcal{X}$. This map

[9] In the sense of [V-sym]

is a section of the projection $\mathcal{X} \to \mathbb{C}P^1$, hence defines a point $\psi(\mathcal{B}) \in \mathrm{Sec}$. The correspondence $\mathcal{B} \mapsto \psi(\mathcal{B})$ comes from a holomorphic map $\psi : \mathcal{M}^s_{\text{fib}} \to \mathrm{Sec}$ of the corresponding moduli spaces.

7.5 Coarse and fine moduli spaces: a digression

In order to prove that the map $\psi : \mathcal{M}^s_{\text{fib}} \to \mathrm{Sec}$ is an isomorphism, we need to make a digression about universal objects and coarse moduli spaces.

Let $\mathfrak{Var}$ be the category of complex-analytic varieties and let $\mathcal{F} : \mathfrak{Var} \to \mathrm{Sets}$ be a functor. Recall that a complex-analytic space Y is said to be a fine moduli space for the functor $\mathcal{F}$ if $\mathcal{F} \cong \mathcal{H}om(\bullet, Y)$. This implies that there exists an element $C \in \mathcal{F}(Y)$ such that for every complex-analytic space U and an element $a \in \mathcal{F}(U)$ there exists a unique map $f : U \to Y$ such that $a = \mathcal{F}(f)(C)$. Such an element C is called *the universal solution* to the moduli problem posed by $\mathcal{F}$.

It is well-known that geometric moduli problems only rarely admit fine moduli spaces. The common way to deal with this is to introduce a weaker notion of a *coarse moduli space*. For the purposes of this paper the following notion suffices.

Definition 7.4 A complex-analytic space Y is called a **coarse moduli space for the problem posed by** $\mathcal{F}$ if for any complex-analytic space Z and an element $a \in \mathcal{F}(Z)$ there exist a unique map $f : Z \to Y$, an open covering U_α of the space Y and a collection $C_\alpha \in \mathcal{F}(U_\alpha)$ such that for every index α

$$\mathcal{F}(f)(C_\alpha) = a|_{f^{-1}(U_\alpha)}.$$

Heuristically, a coarse moduli space Y admits locally a universal solution for the moduli problem $\mathcal{F}$, but these solutions need not come from a single global solution in $\mathcal{F}(Y)$.

All the moduli spaces constructed as infinite-dimensional quotients by means of the slice theorem are coarse moduli spaces in the sense of this definition. This applies to all the moduli spaces considered in this paper, and to the space $\mathcal{X}$ in particular. Therefore for every point $x \in \mathcal{X}$ there exists a neighborhood $U_x \subset X$ and a coherent sheaf $\mathcal{E}_x$ on $U \times X$ which is a family of fiber-supported stable sheaves on X universal for the moduli problem. Let U be such a neighborhood. Then the universality of the sheaf $\mathcal{E}$ implies that

$$\operatorname{Aut} \mathcal{E} = \Gamma(U, \mathcal{O}^*).$$

Lemma 7.5 *Let $\mathbb{C}P^1 \to \mathcal{X}$ be a section of the projection $\pi : \mathcal{X} \to \mathbb{C}P^1$. There exists a coherent sheaf $\mathcal{E}$ on $\mathbb{C}P^1 \times X$ such that for every $x \in \mathbb{C}P^1 \subset \mathcal{X}$ the restriction $\mathcal{E}|_{U_x \times X}$ is isomorphic to the universal sheaf $\mathcal{E}_x$.*

Proof. Indeed, cover $\mathbb{C}P^1$ by open subsets of the form U_x and choose a finite subcovering U_α. In order to define a sheaf $\mathcal{E}$, it is enough to choose a system of isomorphisms

$$g_{\alpha\beta} : \mathcal{E}_\alpha|_{(U_\alpha \cap U_\beta)\times X} \to \mathcal{E}_\beta|_{(U_\alpha \cap U_\beta)\times X}$$

for every intersection $U_\alpha \cap U_\beta$ so that $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ for every three indices α, β, γ. Since $\operatorname{Aut} \mathcal{E}_\alpha = \mathcal{O}^*_{U_\alpha}$, the obstruction to finding such a system of isomorphisms lies in the second Čech cohomology group $H^2(\mathbb{C}P^1, \mathcal{O}^*)$. Consider the long exact sequence

$$H^2(\mathbb{C}P^1, \mathcal{O}) \longrightarrow H^2(\mathbb{C}P^1, \mathcal{O}^*) \longrightarrow H^3(\mathbb{C}P^1, \mathbb{Z})$$

associated to the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0.$$

Since $H^2(\mathbb{C}P^1, \mathcal{O}) = H^3(\mathbb{C}P^1, \mathbb{Z}) = 0$, the group $H^2(\mathbb{C}P^1, \mathcal{O}^*)$ vanishes. ∎

7.6 Conclusion

We can now finish the proof of Theorem 7.2. It remains to prove that the map $\psi : \mathcal{M}^s_{\text{fib}} \to \mathcal{X}$ is an isomorphism. We will construct an inverse map $\psi^{-1} : \mathcal{X} \to \mathcal{M}^s_{\text{fib}}$.

Let $x \in \operatorname{Sec}$ be a point and let $\widetilde{x} : \mathbb{C}P^1 \to \mathcal{X}$ be the corresponding section. Let $\mathcal{E}$ be the coherent sheaf on $\widetilde{x}(\mathbb{C}P^1) \times X$ constructed in Lemma 7.5. Let $\Delta = \pi \times \operatorname{id} : X \hookrightarrow \mathbb{C}P^1 \times X$ be the embedding of X into $\mathbb{C}P^1 \times X$ as the preimage of the diagonal under the natural projection $\operatorname{id} \times \pi : \mathbb{C}P^1 \times \mathcal{X} \to \mathbb{C}P^1 \times \mathbb{C}P^1$. The sheaf $\mathcal{E}$ is by definition isomorphic to the direct image of a holomorphic vector bundle $\mathcal{B}$ on X: $\mathcal{E} \cong \Delta_* \mathcal{B}$.

For every point $I \in \mathbb{C}P^1$ the coherent sheaf $i_* i^* \mathcal{B}$ on X is canonically isomorphic to the restriction of $\mathcal{E}$ to $I \times X \subset \mathbb{C}P^1 \times X$. Therefore the bundle $\mathcal{B}$ is stable by Lemma 7.3. Let $\psi^{-1}(x) \in \mathcal{M}^s_{\text{fib}}$ be the corresponding point in the moduli space $\mathcal{M}^s_{\text{fib}}$.

By construction $\psi(\psi^{-1}(x)) = x$. To prove that $\psi^{-1} \circ \psi = \operatorname{id}$, consider a stable bundle $\mathcal{B} \in \mathcal{M}^s_{\text{fib}}$. Let $\mathcal{E}$ be the coherent sheaf on $\mathbb{C}P^1 \times X$ constructed in Lemma 7.5 and let $p : \mathbb{C}P^1 \times X \to X$ be the projection onto the second factor. By definition

$$\mathcal{E} \cong \Delta_* \Delta^* (p^* \mathcal{B}) \cong \Delta_* (\Delta \circ p)^* \mathcal{B} \cong \Delta_* \mathcal{B}.$$

Therefore $\psi^{-1}(\psi(\mathcal{B})) = \mathcal{B} \in \mathcal{M}^s_{\mathrm{fib}}$. This finishes the proof of Theorem 7.2.

8 Conjectures and open questions.

8.1 NHYM moduli spaces and hyperkähler reduction

8.1.1

Let M be a Kähler manifold and let $\mathcal{M}^s$ be the moduli space of NHYM connections on a complex bundle $\mathcal{B}$ over M. We have shown in Section 2 that the space $\mathcal{M}^s$ is equipped with a natural closed holomorphic 2-form Ω which is is symplectic at least in a neighborhood of the subset of Hermitian connections. In fact one could hope for a much stronger statement.

Conjecture 8.1 *There exists a hyperkähler metric on $\mathcal{M}^s$ such that Ω is the associated holomorphic symplectic from.*

Note that the construction of the form Ω is completely parallel to a construction of a holomorphic symplectic form on the Hitchin-Simpson moduli space $\mathcal{M}^s_{DR}$ of flat connections on $\mathcal{B}$ ([Sim2]). The analog of Conjecture 8.1 for $\mathcal{M}^s_{DR}$ is known.

8.1.2

To provide some evidence for Conjecture 8.1, we give an interpretation of the NHYM equation in the context of hyperkähler reduction.

Let $\mathcal{A}$ be the space of all connections on the complex vector bundle $\mathcal{B}$. The space $\mathcal{A}$ is an affine space over the complex vector space $\mathcal{A}^1(M, \mathfrak{End}\,\mathcal{B})$ of $\mathfrak{End}\,\mathcal{B}$-valued 1-forms on M. Choose an Hermitian metric h on the bundle $\mathcal{B}$. The decomposition

$$\mathcal{A}^1(M, \mathfrak{End}\,\mathcal{B}) = \mathcal{A}^{1,0}(M, \mathfrak{End}\,\mathcal{B}) \oplus \mathcal{A}^{0,1}(M, \mathfrak{End}\,\mathcal{B})$$

allows one to define a quaternionic structure on the space $\mathcal{A}^1(M, \mathfrak{End}\,\mathcal{B})$. Together with the natural trace metric, this structure makes the space $\mathcal{A}$ an (infinite-dimesional) hyperkähler manifold.

The complex gauge group $\mathcal{G} = \mathrm{Maps}(M, \mathrm{Aut}\,\mathcal{B})$ acts on the space $\mathcal{A}$. This action is compatible with the hyperkähler structure on $\mathcal{A}$. Therefore one can apply to the space $\mathcal{A}$ the machinery of **hyperkähler reduction** (see [HKLR]). It turns out that the complex moment map $\mathcal{A} \to \mathbb{C}$ is equal to the map $YM : \mathcal{A} \to \Gamma(M, \mathfrak{End}\,\mathcal{B})$, $\nabla \mapsto \Lambda\nabla^2$. Vanishing of this map is precisely the NHYM condition.

Let $\mathcal{A}_0 = YM^{-1}(0) \subset \mathcal{A}$ be the subset of connections with $\Lambda\nabla^2 = 0$. By the general principles of hyperkähler reduction the quotient $\mathcal{A}_0/\mathcal{G}$ should be hyperkähler. The NHYM moduli space $\mathcal{M}^s$ is the closed subset $\mathcal{M}^s \subset \mathcal{A}_0/\mathcal{G}$ of equivalence classes of connections with curvature $R = \nabla^2$ which satisfy $\Lambda R = 0$ and are, in addition, of Hodge type $(1,1)$. We expect that the embedding $\mathcal{M}^s \hookrightarrow \mathcal{A}_0/\mathcal{G}$ is compatible with the hyperkähler structure on $\mathcal{A}_0/\ G$ and gives a hyperkähler structure on $\mathcal{M}^s$ by restriction.

8.1.3

The hyperkähler reduction construction of the NHYM moduli space also allows to formulate an analog of the Uhlenbeck-Yau Theorem for NHYM-bundles. We first give a new definition of stability for NHYM bundles, more natural than the $(0,1)$-stability used in the body of the paper.

Let $\overline{M}$ be the complex-conjugate complex manifold to M. Since M and $\overline{M}$ are the same as smooth manifolds, the bundle $\mathcal{B}$ can be also considered as a complex vector bundle on $\overline{M}$. For every connection ∇ on $\mathcal{B}$ the $(1,0)$-part $\nabla^{1,0}$ defines a holomorphic structure on the complex bundle $\mathcal{B}$ on $\overline{M}$.

Definition 8.2 Let $\langle\mathcal{B}, \nabla\rangle$ be a bundle with a $(1,1)$-connection over a complex manifold X. Let $U \subset X$ be a Zariski open subset in X, and let $\mathcal{F} \subset \mathcal{B}\big|_U$ be a subbundle which is preserved by ∇. Then $\mathcal{F}$ is called **a subsheaf of** $\langle\mathcal{B}, \nabla\rangle$ if the following two conditions hold:

(i) Consider $\mathcal{B}$ as a holomorphic bundle over M, with a holomorphic structure defined by the $(0,1)$-part of the connection. Then there exist a coherent subsheaf $\widetilde{\mathcal{F}} \subset \mathcal{B}$ on M such that the restriction $\widetilde{\mathcal{F}}\big|_U$ is a sub-bundle of $\mathcal{B}$ which coinsides with $\mathcal{F}$.

(ii) Consider $\mathcal{B}$ as a holomorphic bundle over $\overline{M}$, with a holomorphic structure defined by the $(1,0)$-part of the connection. Then there exist a coherent subsheaf $\widetilde{\mathcal{F}} \subset \mathcal{B}$ on $\overline{M}$ such that the restriction $\widetilde{\mathcal{F}}\big|_U$ is a sub-bundle of $\mathcal{B}$ which coinsides with $\mathcal{F}$.

For a subsheaf $\mathcal{F} \subset \mathcal{B}$, it is straightforward to define the Chern classes and the degree. As usually, $\mathcal{F}$ is called **destabilizing** if

$$\frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}} \geq \frac{\deg \mathcal{B}}{\operatorname{rank} \mathcal{B}}$$

Definition 8.3 Let $\langle \mathcal{B}, \nabla \rangle$ be a bundle with $(1,1)$-connection over a compact Kähler manifold M. Then $\langle \mathcal{B}, \nabla \rangle$ is called ∇-stable if there are no destabilizing subsheaves $\mathcal{F} \subset \langle \mathcal{B}, \nabla \rangle$.

This definition generalizes the definition [Sim3] of stability for flat bundles.

Remark 8.4 Clearly, for NHYM bundles, $(0,1)$-stability implies the stability in the sense of Definition 8.3

An analogy with the Kempf-Ness Theorem suggests that every stable $\mathcal{G}$-orbit in $\mathcal{A}_0$ has non-trivial intersection with the zero set of the real moment map $\mathcal{A}_0 \rightarrow \Gamma(M, \mathfrak{End}_{\mathbb{R}} \mathcal{B})$ from $\mathcal{A}_0$ to the space of anti-Hermitian endomorphisms of the bundle $\mathcal{B}$. This moment map can be described more explicitly.

Definition 8.5 (pseudocurvature) Let $\langle \mathcal{B}, \nabla \rangle$ be a bundle with $(1,1)$-connection and a Hermitian metric h, not necessary compatible. Let $\nabla = \nabla' + \nabla''$ be the decomposition of ∇ onto $(1,0)$ and $(0,1)$-parts. Consider the connection in $\overline{\mathcal{B}}^*$ associated with ∇. Since h identifies $\mathcal{B}$ and $\overline{\mathcal{B}}^*$, this gives another connection in $\mathcal{B}$, denoted by ∇_h. The average $\nabla_h = \frac{\nabla + \nabla_h}{2}$ is again a connection, and is compatible with h. Let θ be the difference $\theta := \frac{\nabla - \nabla_h}{2}$, which is a tensor. Applying ∇_h to θ, we obtain a 2-form Ξ with coefficients in $\mathfrak{End}(\mathcal{B})$. This form Ξ is called **the pseudocurvature** of the triple $\langle \mathcal{B}, \nabla, h \rangle$.

It turns out that the real moment map on a NHYM connection ∇ is given by

$$\nabla \mapsto \Lambda(\Xi),$$

where Ξ is the pseudocurvature.

Definition 8.6 Let $\langle \mathcal{B}, \nabla \rangle$ be a bundle with a NHYM $(1,1)$-connection, and let h be an Hermitian metric on $\mathcal{B}$, not necessarily compatible with ∇ Then h is called **harmonic** if $\Lambda \Xi = 0$, where Ξ is the pseudocurvature of $\langle \mathcal{B}, \nabla, h \rangle$.

Conjecture 8.7 *Let $\langle \mathcal{B}, \nabla \rangle$ be a bundle with NHYM connection ∇. Then there exists a harmonic metric h on $\mathcal{B}$ if and only if $\mathcal{B}$ is a direct sum of*

∇-stable bundles. Also, if $\mathcal{B}$ itself is ∇-stable, then h is unique, up to a constant factor.

An analogous statement is known for flat connections. See [Sim3] for a discussion.

8.2 Kähler base manifold: open questions.

In this subsection, we relate questions pertaining to the case of base manifold M compact and Kähler, but not necessarily hyperkähler.

Question 8.8 Let (B, ∇) be a NHYM-connection in a bundle with zero Chern classes. Is it true that ∇ is necessarily flat?

In Hermitian case, the answer is affirmative by Lübcke [Lü] and Simpson [Sim1]. In a neighbourhood of Hermitian Yang-Mills connection, all NHYM connections on a bundle with zero Chern classes are also flat, at one can see, e. g., from Proposition 2.26. The hyperkähler analogue of this question is Question 8.11.

Let B be a stable holomorphic bundle over M and let $St(B)$ be the deformation space of stable holomorphic structures on B. In Section 2, we defined a Kuranishi map $\phi: U \hookrightarrow H^1(\mathfrak{End}(B))$, where U is a neighbourhood of $[B]$ in $St(B)$. The map ϕ is, locally, a closed embedding, and its image in a neighbourhood of zero in $H^1(\mathfrak{End}(B))$ is an algebraic subvariety, defined by the zeroes of so-called Massey products. Let C be the Zariski closure of the image $\phi(U)$ in $H^1(\mathfrak{End}(B))$. Let $\mathrm{NHYM}(B)$ be the space of NHYM connections inducing the same holomorphic structure. In (2.5), we construct the map $\mathrm{NHYM}(B) \xrightarrow{\rho} \overline{C}$, where $\overline{C}$ is a complex conjugate manifold to C, and prove that in a neighbourhood of zero ρ is isomorphism.

Two questions arise:

Question 8.9 Is the map ρ surjective?

Question 8.10 Is the map ρ etale? Bijective?

These two questions might be reformulated in a purely algebraic way. Let B be a stable holomorphic bundle equipped with a Hermitian Yang-Mills metric, and X be the space of all $(1,0)$-forms $\theta \in \Lambda^{1,0}(\mathfrak{End}(B))$ satisfying

$$\begin{cases} \partial\theta &= \theta \wedge \theta \\ \partial^*\theta &= 0, \end{cases} \tag{8.11}$$

where ∂ is the $(1,0)$-part of the connection, and ∂^* the adjoint operator. The middle cohomology space of the complex

$$\Lambda^{2,0}(\mathfrak{End}(B)) \xrightarrow{\partial^*} \Lambda^{1,0}(\mathfrak{End}(B)) \xrightarrow{\partial^*} \mathfrak{End}(B)$$

is naturally isomorphic to the complex conjugate space to $H^1(\mathfrak{End}(B))$. This gives a map

$$X \xrightarrow{\rho} \overline{H^1(\mathfrak{End}(B))},$$

associating to θ its cohomology class. Then, Proposition 2.16 implies that the image of ρ lies in $\overline{C}$ and locally in a neighbourhood of zero, $\rho: X \longrightarrow \overline{C}$ is an isomorphism. Question 8.9 asks whether ρ is surjective onto $\overline{C}$, and Question 8.10 asks whether $\overline{\pi}$ is etale, or even invertible.

8.3 Autodual and NHYM connections over a hyperkähler base.

8.3.1

The first and foremost question (partially answered in Theorem 3.11; see also Question 8.8):

Question 8.11 Let (B,∇) be a NHYM bundle over a hyperkähler base manifold. Is (B,∇) necessarily autodual?

8.3.2

Let M be a compact hyperkähler manifold, and let $\mathcal{S}$ be a connected component of the moduli of autodual connections on a complex vector bundle $\mathcal{B}$. Assume that $\mathcal{S}$ contains a point B which is Hermitian autodual. Consider the "Mukai dual" space $\widehat{M}$, that is, the moduli space of Hermitian autodual connections on $\mathcal{B}$ (Subsection 7.1). Assume that the connected component of $\widehat{M}$ containing B is smooth and compact. Clearly, then, all connections from $\mathcal{S}$ are fiberwise stable, in the sense of Definition 7.1. Thus, Theorem 7.2 gives an isomorphism between $\mathcal{S}$ and the space $Sec(\widehat{M})$ of twistor lines in $\operatorname{Tw}(\widehat{M})$.

In such situation, we are going to give a conjectural description of the space $\mathcal{S}$, assuming that the answer to 8.9—8.10 is affirmative.

8.3.3

Definition 8.12 (Twisted cotangent bundle) Let M be a Kähler manifold, $\Omega^1 M$ its holomorphic cotangent bundle. The Kähler class

$$\omega \in H^1(\Omega^1 M) = Ext^1(\mathcal{O}(M), \Omega^1 M)$$

gives by Yoneda an exact sequence

$$0 \longrightarrow \Omega^1 M \longrightarrow E \stackrel{e}{\longrightarrow} \mathcal{O}(M) \longrightarrow 0,$$

where $\mathcal{O}(M)$ is the trivial one-dimensional bundle. Let ν be a non-zero section of $\mathcal{O}(M)$, and E_ν be the set of all vectors

$$\left\{ v \in E\big|_m \;\middle|\; e(v) = \nu\big|_m \right\}$$

where m runs through all points of M. Consider E_ν as a submanifold in the total space of E. Then E_ν is called **a twisted cotangent bundle of** M, denoted by $\Omega_\omega M$.

The space $\Omega_\omega M$ has a natural action of $\Omega^1 M$ considered as a group scheme over M, and as such is a torsor over $\Omega^1 M$

8.3.4

The affirmative answer to the stronger form of 8.10 would give the proof of the following conjecture.

Conjecture 8.13 *Under assumptions of 8.3.2, there exists a natural isomorphism of complex manifolds*

$$Sec(\widehat{M}) \cong \Omega_\omega M,$$

where $\Omega_\omega M$ is the twisted cotangent bundle.

In the general situation, there is a natural map from the space of twistor lines $Sec(M)$ of a compact hyperkähler manifold to $\Omega_\omega M$. However, in general there are no approaches to the proof of surjectivity.

Example 8.14 Let M be a compact complex torus, $\dim_{\mathbb{C}} M = 2n$, and B a trivial line bundle. Clearly, M is hyperkähler. Then $\widehat{M}$ is the dual torus, and $\mathcal{S}$ is the space of local systems on M, which is isomorphic to $(\mathbb{C}^*)^{2n}$. The space C of 8.10 is isomorphic to $H^1(\mathcal{O}M)$, and the answer to 8.10 is obviously affirmative. Thus, $Sec(M)$ and $\Omega_\omega M$ are also isomorphic to $(\mathbb{C}^*)^{2n}$ and are Stein.

In the following subsection, we shall see that this is indeed a general phenomenon – the space of twistor lines is equipped with a canonical plurisubharmonic function and is likely to be Stein. However, we don't know a general argument constructing plurisubharmonic functions on the twisted cotangent bundle – this is one more mystery.

8.4 Plurisubharmonic functions on moduli spaces.

Let M be a hyperkähler manifold $\mathrm{Tw} \xrightarrow{\pi} \mathbb{C}P^1$ its twistor space, and $Sec(M)$ the space of sections $s : \mathbb{C}P^1 \longrightarrow \mathrm{Tw}$ of the map π, also called *twistor lines* (Section 4). It is easy to equip $Sec(M)$ with a natural plurisubharmonic function.

Recall that Tw is isomorphic as a C^∞-manifold to $M \times \mathbb{C}P^1$. This decomposition gives a natural (non-Kähler) Hermitian metric on Tw.

Proposition 8.15 *Consider the function $v : Sec(M) \longrightarrow \mathbb{R}^+$ which maps a line $s \in \mathrm{Tw}$ to its Hermitian volume, taken with respect to the Hermitian metric on* Tw. *Then v is strictly plurisubharmonic.*

Proof. Let ω be the differential 2-form which is the symplectic part of the Hermitian metric on Tw. Since the twistor lines are complex subvarieties in Tw, $v(s) = \int_s \omega$ for all twistor lines s. Then, for all bivectors $x, \bar{x}$ in $T_s Sec(M)$, we have

$$\partial\bar{\partial} v(x, \bar{x}) = \int_s \partial\bar{\partial}\omega(\mathbf{x}, \bar{\mathbf{x}}), \tag{8.12}$$

where $\mathbf{x}, \bar{\mathbf{x}}$ are the sections of $T\,\mathrm{Tw}\big|_s$ corresponding to $x, \bar{x}$. Then, to prove that v is plurisubharmonic it suffices to show that $\partial\bar{\partial}\omega(\mathbf{x}, \bar{\mathbf{x}})$ is positive. From Lemma 4.4, it is easy to see that $\partial\bar{\partial}\omega = \omega \wedge \pi^* \mathrm{FS}(\mathbb{C}P^1)$, where $\mathrm{FS}(\mathbb{C}P^1)$ is the Fubini-Study form on $\mathbb{C}P^1$. Clearly, then, $\partial\bar{\partial}\omega(\mathbf{x}, \bar{\mathbf{x}})$ is positive, and v is plurisubharmonic. This proves Proposition 8.15. ∎

One of the most intriguing questions of hyperkähler geometry is to learn whether the function v is exhausting.

In notation and assumptions of 8.13, consider the space $Sec(\widehat{M})$ which is isomorphic to the space $\mathcal{S}$ of autodual connections. There is the canonical Weil-Petersson metric on $\mathcal{S}$, coming from results of Subsection 8.1. This metric is hyperkähler. This metric is given by a potential, which is equal to the integral of the square of the absolute value of the curvature.

Question 8.16 Is the Weil–Petersson metric related to the metric given by v?

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